## 8.   On Uniqueness and Estimations for Solutions of Modified Frankl' Problem for Linear and Nonlinear Equations of Mixed Type

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**1. Introduction.**   Concerning the Frankl' problem for equations of mixed type which has been proposed by F. I. Frankl' (see [3] and references quoted there), the uniqueness and the maximum principles are discussed under some restrictions by many authors in USSR using the method of singular integral equations or the abc method.   In the present paper, with the intention of using Agmon, Nirenberg and Protter type maximum principle [1] we slightly modify the boundary condition of the problem on some hyperbolic boundary in such a way as the directional derivative of the solution in the direction of a characteristic is given, while in the Frankl's original problem the derivative with respect to $x$ is given.   Then we prove the uniqueness and lead some estimations for the solutions of linear and nonlinear problems. On the basis of the above modification, we infer that we shall be able to return to the discussion of the original problem, e.g. the uniqueness of the solution and the existence of a weak solution, but these matters will be discussed elsewhere.

**2. Definitions and problems.**   Let $K(y)$ be a function of $y$ defined and twice continuously differentiable on an interval $(-y_1, y_2)$ where $y_1$, $y_2 > 0$, and which has the property $yK(y) > 0$ for $y \neq 0$.

We shall define a domain $\Omega$ in the $x, y$-plane satisfying the condition that the ordinates of the points of the closure $\bar{\Omega}$ are contained in the interval $(-y_1, y_2)$ as follows.   Let us take two points $A(a, 0)$ and $B(b, 0)$ on the $x$-axis with $a < b$ and let $C$ be the intersection point of two curves in $y < 0$, one issuing from $A$ has the slope $0 \geq dx/dy > -\sqrt{-K(y)}$ and the other issuing from $B$ has the slope $0 \leq dx/dy < \sqrt{K(y)}$. We shall denote the arcs $AC$ and $BC$ by $\gamma_1$ and $\gamma_2$, respectively.   Further, let $D(d, 0)$ and $E(e, 0)$ be two points on the $x$-axis with $d < a, e > b$ and let $\sigma$ be a Jordan arc in $y > 0$ joining $D$ and $E$ where it is assumed that the length of $\sigma$ is not less than the length $l$ of $\gamma_1$.   Let $F$ be a point on $\sigma$ such that the length of the arc $DF$ denoted by $\sigma_0$ equals $l$.   $\Omega$ shall be the domain enclosed with the curve $ACBEFDA$.   Let $\Omega_1 = \Omega \cap \{y > 0\}$ and $\Omega_2 = \Omega \cap \{y < 0\}$.

In $\Omega$, we shall consider the following differential operators:

( 1 )                $Tu=K(y)u_{xx}+u_{yy}$,

( 2 )            $Lu=Tu+a(x, y)u_x+b(x, y)u_y+c(x, y)u$,

where $a(x, y)$, $b(x, y) \in C^1(\Omega) \cap C^0(\overline{\Omega})$ and $c(x, y) \in C^0(\overline{\Omega})$.

In $\overline{\Omega}_2$, we shall define two directional derivatives $v_\xi = \sqrt{-K(y)}v_x + v_y$ and $v_\eta = -\sqrt{-K(y)}v_x + v_y$ for a function $v(x, y)$ in the directions of the characteristics $dx/dy = \sqrt{-K(y)}$ and $dx/dy = -\sqrt{-K(y)}$, respectively. Let $v_\alpha$ on $\sigma_0$[1] denote the directional derivative for a function $v(x, y)$ in the direction of the vector $\alpha$ whose inner product with the inner normal vector to $\sigma_0$ is positive.

**Definition.** Let us consider the following six functions defined and continuous on each part of the boundary of $\Omega$:

$\phi_1(x, y)$ on $\overline{\sigma \setminus \sigma_0}$, $\phi_2(x)$ on $\overline{DA}$, $\phi_3(x)$ on $\overline{BE}$,

$\phi_4(x, y)$ on $\overline{\gamma}_1$, $\phi_5(x, y)$ and $\phi_6(x, y)$ on $\overline{\sigma}_0$,

where $\phi_3(e) = \phi_1(e, 0)$ and $\phi_2(d) - \phi_2(a) = \phi_5(d, 0)$. We shall say a function $u(x, y)$ defined on $\overline{\Omega}$ satisfies the boundary condition $u \in B(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6)$, if it satisfies the relations

( 3 )  $\begin{cases} u(x, y)=\phi_1(x, y) \text{ on } \overline{\sigma \setminus \sigma_0}, u(x, 0)=\phi_2(x) \text{ on } \overline{DA}, \\ u(x, 0)=\phi_3(x) \text{ on } \overline{BE}, u_\eta(x, y)=\phi_4(x, y) \text{ on } \gamma_1, \\ u(x, y)-u(X, Y)=\phi_5(x, y) \text{ for } (x, y) \in \overline{\sigma}_0 \text{ and} \\ (X, Y) \in \overline{\gamma}_1 \text{ and } u_\alpha(x, y)=\phi_6(x, y) \text{ on } \sigma_0, \end{cases}$

where in the fifth relation $(X, Y) \in \overline{\gamma}_1$ corresponds to $(x, y) \in \overline{\sigma}_0$ in such a way that the length of the arc from $A$ to $(X, Y)$ is equal to the length of the arc from $D$ to $(x, y)$.

We shall consider a set of functions $u(x, y)$ which are defined on $\overline{\Omega}$, belong to $C^2(\Omega) \cap C^0(\overline{\Omega})$, have the directional derivatives $u_\alpha$ on $\sigma_0$ and have the directional derivatives $u_\eta$ which are continuous up to $\gamma_1$ inclusive. Let us denote temporarily such a set by $\tilde{C}^2(\overline{\Omega})$.

**Problem I.** To seek a function $u(x, y)$ belonging to $\tilde{C}^2(\overline{\Omega})$ which satisfies the equation

( 4 )                    $Lu=f(x, y)$

in $\Omega$ and the boundary condition

( 5 )            $u \in B(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6)$,

where $f(x, y)$ is an arbitrary continuous function defined on $\overline{\Omega}$. Such a function $u(x, y)$ is called "regular solution of Problem I".

**Problem II.** To seek a function $u(x, y)$ belonging to $\tilde{C}^2(\overline{\Omega})$ which satisfies the equation

( 6 )            $Tu=f(x, y, u, u_x, u_y)$

in $\Omega$ and the boundary condition (5), where $f(x, y, z, p, q)$ is an arbitrary continuous function defined on $\overline{\Omega} \times R^3$. Such a function $u(x, y)$ is

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1) In what follows the arc or interval with a bar contains its end points, but the one without a bar does not.

called "regular solution of Problem II".

**Definition.** When the following relations hold for the coefficients $K, a, b, c$ we say that the operator $L$ satisfies the condition (A):

$$(A) \quad \begin{cases} c \leq 0 \text{ in } \Omega, \\ a + b\sqrt{-K} + (\sqrt{-K})_y < 0 \text{ in } \Omega_2, \\ 4(-K)c + [a - b\sqrt{-K} + 3(\sqrt{-K})_y] \cdot [a + b\sqrt{-K} + (\sqrt{-K})_y] \\ \quad - 2\sqrt{-K}[a + b\sqrt{-K} + (\sqrt{-K})_y]_\xi \geq 0 \text{ in } \Omega_2. \end{cases}$$

**Definition.** When the following relations hold for the function $K$ and the derivatives of $f(x, y, z, p, q)$, we say that the operator $T$ and the function $f$ satisfy the condition (B):

$$(B) \quad \begin{cases} f_z \geq 0 \text{ in } \Omega \times R^3, \\ f_p + f_q\sqrt{-K} - (\sqrt{-K})_y > 0 \text{ in } \Omega_2 \times R^3, \\ 4(-K)f_z + [-f_p + f_q\sqrt{-K} + 3(\sqrt{-K})_y] \cdot [f_p + f_q\sqrt{-K} - (\sqrt{-K})_y] \\ \quad - 2\sqrt{-K}[f_p + f_q\sqrt{-K} - (\sqrt{-K})_y]_\xi \leq 0 \text{ in } \Omega_2 \times R^3. \end{cases}$$

**3. Maximum principle. Theorem 1.** *Let the operator $L$ satisfy the condition* (A). *If a function $u(x, y)$ which is defined on $\bar{\Omega}$ and belongs to $\tilde{C}^2(\bar{\Omega})$ satisfies the inequalities $Lu \geq 0$ in $\Omega$, $u_\eta \geq 0$ on $\gamma_1$, $u(x, y) \geq u(X, Y)$ where $(x, y) \in \bar{\sigma}_0$ and $(X, Y) \in \bar{\gamma}_1$ are corresponding points and $u_\alpha \geq 0$ on $\sigma_0$, then the positive maximum value of $u$ in $\bar{\Omega}$ cannot be attained except on $\overline{DA} \cup \overline{BE} \cup \overline{\sigma \backslash \sigma_0}$.*

To prove this theorem, we need the following lemmas, which may be proved in a similar manner as in Agmon, Nirenberg and Protter [1], Oleinik [4] and Protter and Weinberger [5].

**Lemma 1.** *Let the operator $L$ satisfy the condition* (A) *and let us consider a function $u(x, y)$ which is defined on $\bar{\Omega}_2$, belongs to $C^2(\Omega_2) \cap C^0(\bar{\Omega}_2)$ and has the directional derivative $u_\eta$ which is continuous up to $\gamma_1$ inclusive. If the function $u(x, y)$ satisfies the inequalities $Lu \geq 0$ in $\Omega_2$ and $u_\eta \geq 0$ on $\gamma_1$, then the positive maximum value of $u$ in $\bar{\Omega}_2$ cannot be attained except on $\bar{\gamma}_1 \cup \overline{AB}$. Moreover, if the value is attained at some point $(x_0, 0) \in AB$, then we have*

$$\liminf_{y \to -0} \frac{u(x_0, y) - u(x_0, 0)}{y} > 0.$$

**Lemma 2.** *Let a function $u(x, y)$ which is defined on $\bar{\Omega}_1$ and belongs to $C^2(\Omega_1) \cap C^0(\bar{\Omega}_1)$ satisfy the inequality $Lu \geq 0$ in $\Omega_1$. Assume $c \leq 0$ in $\Omega_1$. Then the positive maximum value of $u$ in $\bar{\Omega}_1$ cannot be attained at any point in the interior of $\Omega_1$. Moreover, if the value is attained at some point $(x_0, y_0) \in \sigma_0 \cup AB$, then we have*

$$\limsup_{a \to +0} \frac{u(x_0 + a\kappa_1, y_0 + a\kappa_2) - u(x_0, y_0)}{a|\kappa|} < 0,$$

*where the inner product of the vector $\kappa = (\kappa_1, \kappa_2)$ and the inner normal vector to $\sigma_0$ or $AB$ at $(x_0, y_0)$ is positive.*

**Proof of Theorem 1.** By virtue of Lemma 1 the positive maxi-

mum value of $u$ in $\bar{\Omega}$ is not attained at an interior point of $\Omega_2$ and on $\gamma_2$, and by Lemma 2 also not at an interior point of $\Omega_1$. Therefore it is attained at a point of the boundary of $\Omega$ except $\gamma_2$ or at a point of $\overline{AB}$. The case when the point lies on $AB$ is impossible to occur by Lemmas 1 and 2. If the maximum point lies on $\gamma_1$, the maximum must be attained at a corresponding point on $\sigma_0$ from the assumption of the theorem, and then at that point we must have $u_\alpha < 0$ owing to Lemma 2. This contradicts with the assumption of the theorem. Thus the maximum cannot be attained on $\gamma_1$ and $\sigma_0$. This completes the proof.

**Remark 1.** If $c \equiv 0$, the results of Theorem 1 and Lemmas 1, 2 hold without the assumption of positivity of the maximum value.

**4. Uniqueness and estimations.** Using Theorem 1 we have the following theorems.

**Theorem 2.** *If the operator $L$ satisfies the condition* (A), *Problem I has at most one solution.*

**Theorem 3.** *Let the operator $L$ satisfy the condition* (A) *and moreover have the property $c \leq -k^2 < 0$ with a constant $k$. If the function $u(x, y)$ is a regular solution of Problem I, there holds the estimation*

$$
\begin{aligned}
(7) \quad |u| \leq &\max |\phi_1| + \max |\phi_2| + \max |\phi_3| \\
&+ C_1(\max |\phi_4| + \max |\phi_5| + \max |\phi_6|)\{4 + (C_2/k^2) \max [|K| \\
&+ |a| + |b| + |c| + 1]\} + (1/k^2) \max |f| \quad in \ \Omega,
\end{aligned}
$$

*where the constants $C_1$ and $C_2$ are independent of the coefficients in $L$ and the boundary functions. Here it is required that for arbitrary continuous functions $\psi_4(x, y)$ on $\bar{\gamma}_1$, $\psi_5(x, y)$ on $\bar{\sigma}_0$ and $\psi_6(x, y)$ on $\bar{\sigma}_0$, the problem $Lu = 0$ with $u \in B(0, 0, 0, \psi_4, \psi_5, \psi_6)$ has a regular solution.*

**Theorem 4.** *Let the operator $L$ satisfy the condition* (A). *If the function $u(x, y)$ is a regular solution of Problem I, there holds the estimation*

$$
\begin{aligned}
(8) \quad |u| \leq &\max |\phi_1| + \max |\phi_2| + \max |\phi_3| + C_3(\max |\phi_4| + \max |\phi_5| \\
&+ \max |\phi_6|) + C_4 \max |f| \quad in \ \Omega,
\end{aligned}
$$

*where the constants $C_3$ and $C_4$ are independent of the coefficients in $L$ and the boundary functions. Here it is required that for arbitrary continuous functions $\chi(x, y)$ on $\Omega$, $\psi_4(x, y)$ on $\bar{\gamma}_1$, $\psi_5(x, y)$ on $\bar{\sigma}_0$ and $\psi_6(x, y)$ on $\bar{\sigma}_0$ there exists a regular solution of the problem $Lu = \chi$ with $u \in B(0, 0, 0, \psi_4, \psi_5, \psi_6)$.*

**Theorem 5.** *Let the assumption of Theorem 3 be satisfied. Then a solution $u$ of Problem I which has a bounded second derivative $u_{xx}$ and bounded first derivatives $u_x$, $u_y$ on $\bar{\Omega}$ depends continuously on the coefficients of $L$, the function $f$ and the boundary functions.*

**Theorem 6.** *Let the operator $T$ and the function $f(x, y, z, p, q) \in C^2(\Omega \times R^3)$ satisfy the condition* (B). *Then Problem II has at most one*

*solution.*

**Theorem 7.** *Let the operator $T$ and the function $f(x, y, z, p, q) \in C^2(\Omega \times R^3)$ satisfy the condition (B). Assume that $|f_z|$, $|f_p|$, $|f_q|$ are bounded on $\Omega \times R^3$ and $f_z \geq k^2 > 0$ on $\Omega \times R^3$ for some constant $k$. Then if the function $u(x, y)$ is a solution of Problem II, there holds the estimation*

$$
\begin{aligned}
(9) \quad |u| \leq &\max |\phi_1| + \max |\phi_2| + \max |\phi_3| \\
&+ C_5(\max |\phi_4| + \max |\phi_5| + \max |\phi_6|)\{4 + (C_6/k^2) \max [|K| + 1]\} \\
&+ (1/k^2) \max |f(x, y, 0, 0, 0)| \quad \text{in } \Omega,
\end{aligned}
$$

*where the constant $C_5$ is independent of $K$, $f$ and the boundary functions, and $C_6$ depends only on the bounds of the derivatives of $f$. Here it is required that there exists a regular solution of the problem mentioned in Theorem 3 for the operator $L$ having $\hat{A}$, $\hat{B}$ and $\hat{C}$ for $a$, $b$ and $c$, respectively, where*

$$
\hat{A} = -\int_0^1 f_p(x, y, tu, tu_x, tu_y) dt, \quad \hat{B} = -\int_0^1 f_q(\cdots) dt, \quad \hat{C} = -\int_0^1 f_z(\cdots) dt.
$$

**Theorem 8.** *Let the operator $T$ and the function $f(x, y, z, p, q) \in C^2(\Omega \times R^3)$ satisfy the condition (B). Assume that $|f_z|$, $|f_p|$, $|f_q|$ are bounded on $\Omega \times R^3$. Then, if the function $u(x, y)$ is a solution of Problem II, there holds the estimation*

$$
\begin{aligned}
(10) \quad |u| \leq &\max |\phi_1| + \max |\phi_2| + \max |\phi_3| + C_7(\max |\phi_4| + \max |\phi_5| \\
&+ \max |\phi_6|) + C_8 \max |f(x, y, 0, 0, 0)| \quad \text{in } \Omega,
\end{aligned}
$$

*where the constants $C_7$ and $C_8$ are independent of $K$, $f$ and the boundary functions. Here it is required that there exists a regular solution of the problem mentioned in Theorem 4 for $L$ with the coefficients $\hat{A}$, $\hat{B}$, and $\hat{C}$ described in Theorem 7.*

**Remark 2.** The results in this paper remain valid in each case of the modifications (i) the alteration of the order of the points $A$ and $D$ or $B$ and $E$, i.e. $a \leq d$ or $b \geq e$, (ii) the alteration of the definition of the arc $\gamma_2$ such that it has the slope $dx/dy = \sqrt{-K(y)}$ except near the point $C$ and (iii) the alteration of the definition of the arc $\gamma_1$ such that it has the slope $dx/dy = -\sqrt{-K(y)}$, which appear in the Tricomi boundary condition.

## References

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