7. *The Powers of an Operator of Class C_ρ*

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1. In a recent paper [4], M. J. Crabb gives the best bound $\sqrt{2}$ of the inequality proposed by C. A. Berger and J. G. Stampfli [2]:

$$\limsup_{n\to\infty} \|T^n x\| \leqq \sqrt{2}\, \|x\|,$$

for an operator T with $w(T)=1$, where $w(T)$ is the numerical radius of T given by

$$w(T)=\sup\{|(Tx, x)|\,;\, \|x\|=1\}.$$

Using his method, he proves also a generalization of a theorem of Berger-Stampfli [3] and Williams-Crimmins [6]. In the present note, we shall give a further generalization of Crabb's theorem in an elementary method basing on an idea of C. A. Berger and J. G. Stampfli.

2. Following after B. Sz. Nagy and C. Foiaş [5], let C_ρ be the set of all operators acting on a Hilbert space $\mathfrak{H}$ such that there exist a Hilbert space $\mathfrak{K}$ containing $\mathfrak{H}$ as a subspace and a unitary operator U acting on $\mathfrak{K}$ satisfying

$$T^m = \rho P U^m | \mathfrak{H} \qquad (m=1, 2, \cdots), \tag{1}$$

where P is the projection of $\mathfrak{K}$ onto $\mathfrak{H}$. (1) implies at once

$$T^{*m} = \rho P U^{*m} | \mathfrak{H} \qquad (m=1, 2, \cdots). \tag{2}$$

It is well-known by [5] that

$$C_1 = \{T \in \boldsymbol{B}(\mathfrak{H})\,;\, \|T\| \leqq 1\}$$

and

$$C_2 = \{T \in \boldsymbol{B}(\mathfrak{H})\,;\, w(T) \leqq 1\}.$$

Therefore, the following theorem contains Crabb's theorem as a special case ($\rho=2$):

Theorem. *Suppose that $T \in C_\rho (\rho \neq 1)$ and that*

$$\|T^n x\| = \rho \tag{3}$$

for some integer n and a unit vector x. Then we have

(i) $T^{n+1}x=0$,

(ii) $\|T^k x\| = \sqrt{\rho}$ *for* $k=1, 2, \cdots, n-1$,

(iii) $x, Tx, \cdots, T^n x$ *are mutually orthogonal,*

and

(iv) *The linear span $\mathfrak{L}$ of $x, Tx, \cdots, T^n x$ is a reducing subspace of T.*

3. **Proof.** Ad (i). Let T be as in (1). Then

$$\rho\|x\| = \|T^n x\| = \|\rho P U^n x\| = \rho \|P U^n x\|.$$

Since U is unitary and P is a projection, we have

(4) $$PU^n x = U^n x,$$

or $U^n x \in \mathfrak{H}$. Hence

$$T^n x = \rho U^n x.$$

Therefore, we have

$$\rho P U^{n+1} x = T^{n+1} x = T(\rho U^n x) = (\rho P U)(\rho U^n x) = \rho^2 P U^{n+1} x.$$

Hence we have $T^{n+1} x = 0$ for $\rho \neq 1$.

Ad (ii). For each k $(1 \leqq k < n)$, we have

$$\begin{aligned} \|T^k x\|^2 &= (T^k x, T^k x) = \rho^2 (PU^k x, PU^k x) \\ &= \rho^2 (U^{n-k} P U^k x, U^n x) \\ &= \rho^2 (P U^{n-k} P U^k x, U^n x) \\ &= (T^n x, U^n x) \\ &= \rho (P U^n x, U^n x) \\ &= \rho \|U^n x\|^2 = \rho \|x\|^2. \end{aligned}$$

Hence $\|T^k x\| = \sqrt{\rho}$.

Ad (iii). Since $T^{n+j} x = 0$ by (i), we have

$$\begin{aligned} (T^i x, T^j x) &= \rho^2 (P U^i x, U^j x) \\ &= \rho^2 (U^{n-j} P U^i x, U^n x) \\ &= \rho^2 (P U^{n-j} P U^i x, U^n x) \\ &= (T^{n+i-j} x, U^n x) = 0, \end{aligned}$$

for every i and j such as $0 \leqq j < i \leqq n$.

Ad (iv). It is clear that $\mathfrak{L}$ is invariant under T. Therefore it suffices to prove that the vectors $x, Tx, \cdots, T^n x$ are orthogonal to Ta, where a is a vector in $\mathfrak{H}$ which is orthogonal to $\mathfrak{L}$.

For each $k (1 \leqq k \leqq n)$, we have

$$\begin{aligned} (Ta, T^k x) &= (\rho P U a, \rho P U^k x) \\ &= \rho^2 (P U a, U^k x) \\ &= \rho^2 (U^{n-k} P U a, U^n x) \\ &= (T^{n-k+1} a, U^n x) \\ &= \rho (P U^{n-k+1} a, U^n x) \\ &= \rho (U^{n-k+1} a, U^n x) \\ &= \rho (a, U^{k-1} x) \\ &= \rho (a, P U^{k-1} x) \\ &= (a, T^{k-1} x) = 0. \end{aligned}$$

This shows that Ta is orthogonal to $Tx, \cdots, T^n x$. At this end, we shall show that Ta is orthogonal to x. Now, we have

$$\begin{aligned} \|T^{*n} U^n x\| &= \rho \|P U^{*n} U^n x\| \\ &= \rho \|Px\| = \rho \|x\| = \rho. \end{aligned}$$

As $\|x\| = \|U^n x\| = 1$, by (i) we have $T^{*(n+1)} U^n x = 0$. Therefore

$$\begin{aligned} T^{*(n+1)} U^n x &= \rho P U^{*(n+1)} U^n x \\ &= \rho P U^* x = T^* x = 0. \end{aligned}$$

Hence we have finally

$$(Ta, x)=(a, T^*x)=0.$$

This completes the proof.

References

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