6. *Perfect Class of Spaces*

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The author introduced in [6] the notion of perfect class of spaces and showed that the class of ν-spaces is perfect. Recall that a class $\mathfrak{C}$ of spaces is said to be perfect if the following five conditions are satisfied.

(1) If $X \in \mathfrak{C}$, then X is normal.

(2) If $X \in \mathfrak{C}$ and $Y \subset X$, then $Y \in \mathfrak{C}$.

(3) If $X_i \in \mathfrak{C}$, $i=1, 2, \cdots$, then $\Pi X_i \in \mathfrak{C}$.

(4) If $X \in \mathfrak{C}$, then there exists $Z \in \mathfrak{C}$ with $\dim Z \leqq 0$ such that X is the image of Z under a perfect mapping.

(5) If $X \in \mathfrak{C}$ and Y is the image of X under a perfect mapping, then $Y \in \mathfrak{C}$.

It is to be noted that the first three conditions imply that each element of $\mathfrak{C}$ is perfectly normal. The aim of this paper is to show the existence of the maximal perfect subclass in the class of paracompact σ-spaces. A characterization theorem of dimension of cubic μ-spaces will also be stated. All spaces in this paper are assumed to be Hausdorff and all mappings to be continuous. The suffix i runs through the positive integers. Definitions for undefined terminologies can be seen in [6]. The discussion with Professor K. Morita at Shuzenji Hot Spring Symposium, 1970, was suggestive to the present study.

Lemma 1. *If X is a paracompact Σ-space with* $\dim X=0$ *and Y is a paracompact Morita space with* $\dim Y=0$, *then* $\dim (X \times Y)=0$.

This can be proved by almost the same way as in the proof of [3, Theorem 3].

Lemma 2 ([1, Theorem 4]). *Let X be the inverse limit of $\{X_i, \pi^i{}_j\}$, where each X_i is a normal space with* $\dim X_i \leqq n$ *and each $\pi^i{}_j$ is open. If X is countably paracompact, then X is a normal space with* $\dim X \leqq n$.

Lemma 3. *Let X_i, $i=1, 2, \cdots$, be paracompact Σ-spaces with* $\dim X_i=0$. *Then* $\dim (\prod X_i)=0$.

Proof. Since a Σ-space is a Morita space by [2, Theorem 2.7], $\dim (X_1 \times X_2)=0$ by Lemma 1. Let $\prod_{i \leqq j} X_i$, $j>2$, be an arbitrary finite product. Since $\prod_{i<j} X_i$ is a paracompact Σ-space by [2, Theorem 3.13], we can prove easily $\dim (\prod_{i \leqq j} X_i)=0$ by induction with the aid of Lemma 1. Since the infinite product $\prod X_i$ is paracompact by [2, Theorem 3.13], then $\dim (\prod X_i)=0$ by Lemma 2. The proof is finished.

Theorem 1. *Let $\mathfrak{C}$ be the class of all 0-dimensional paracompact σ-spaces, their perfect images and the empty set. Then $\mathfrak{C}$ is perfect.*

Proof. The condition (3) is non-trivial, while the other four conditions are almost evident to be true by the definition of $\mathfrak{C}$. To check (3) let $X_i \in \mathfrak{C}$, $i=1, 2, \cdots$. Let Z_i be a paracompact σ-space with $\dim Z_i \leqq 0$ such that X_i is the image of Z_i under a perfect mapping f_i. Then $\prod X_i$ is the image of $\prod Z_i$ under the perfect mapping $\prod f_i$. Since $\dim (\prod Z_i) \leqq 0$ by Lemma 3, $\prod X_i \in \mathfrak{C}$ and the proof is finished.

Obviously the class of ν-spaces in [6] is a subclass of the above $\mathfrak{C}$. The author does not know whether these two classes are distinct.

Lemma 4. *Let X and Y be paracompact σ-spaces with* $\dim X \leqq n$ *and f a perfect mapping of X onto Y. If for each point y of Y, $f^{-1}(y)$ consists of exactly $k(<\infty)$ points, then* $\dim Y \leqq n$.

Proof. Let $\mathfrak{F}_i$, $i=1, 2, \cdots$, be locally finite closed collections of X such that $\bigcup \mathfrak{F}_i$ forms a network of X and $\mathfrak{F}_i \subset \mathfrak{F}_{i+1}$ for each i. Since $f(\mathfrak{F}_i)$ is point-finite and closure-preserving, it is locally finite in Y. Let $\mathfrak{H}_i$ be the collection of all sets of type $\bigcap_{j=1}^k f(F_j)$ such that $F_j \cap F_m = \emptyset$ whenever $1 \leqq j < m \leqq k$ and $\{F_1, \cdots, F_k\} \subset \mathfrak{F}_i$. If we set $H = \bigcap_{j=1}^k f(F_j)$, then $f | F_j \cap f^{-1}(H)$ is a homeomorphism of $F_j \cap f^{-1}(H)$ onto H. Hence $\dim H \leqq \dim X \leqq n$. Let H_i be the sum of all elements of $\mathfrak{H}_i$. Then $\dim H_i \leqq n$, since $\mathfrak{H}_i$ is a locally finite closed collection of Y.

Let y be an arbitrary point of Y and $\{x_1, \cdots, x_k\}$ be the inverse image of y under f. Then there exist $\mathfrak{F}_m$ and elements $F_1, \cdots, F_k$ of $\mathfrak{F}_m$ such that $x_j \in F_j$ for $j=1, \cdots, k$ and $F_j \cap F_m = \emptyset$ whenever $1 \leqq j < m \leqq k$. Thus $y \in \bigcap_{j=1}^k F_j \in \mathfrak{H}_m$. This implies that $Y = \cup H_i$ and hence $\dim Y = \max \dim H_i \leqq n$. The proof is finished.

As for the definition of a replica of a σ-metric space in the following, see [5].

Lemma 5. *Consider the diagram:*

$$\begin{array}{ccc} Z & \xrightarrow{\ f\ } & X \\ {\scriptstyle\sigma}\big\downarrow & & \big\downarrow{\scriptstyle\rho} \\ \sigma Z & \xrightarrow{\ \hat{f}\ } & \rho X \end{array}$$

Let X be a paracompact σ-metric space, ρX its replica, $\rho: X \to \rho X$ the identity mapping, σZ a metric space and $\hat{f}: \sigma Z \to \rho X$ a perfect mapping onto. Let the set Z be identical with σZ, σ the identity transformation of Z onto σZ and $f: Z \to X$ the transformation such that $\hat{f}\sigma = \rho f$. Let $\mathfrak{U}$ be the topology of X and $\mathfrak{V}$ the topology of σZ. Then Z with the base $f^{-1}(\mathfrak{U}) \wedge \sigma^{-1}(\mathfrak{V})$ is a paracompact σ-metric space such that σZ is a replica of Z and f is a perfect mapping.

This is essentially proved in [5, Theorem 6]. A space X is said to be a cubic μ-space if $X = \prod X_i$, where each X_i is a paracompact σ-

metric space.

Theorem 2. *Let X be a cubic μ-space. Then the following four conditions are equivalent.*

i) $\dim X \leqq n$.

ii) *X is the image of a μ-space Z with* $\dim Z \leqq 0$ *under a perfect mapping of order* $\leqq n+1$.

iii) *X is the sum of $n+1$ subsets H_i, $i=1,\cdots,n+1$, with* $\dim H_i \leqq 0$.

iv) $\operatorname{Ind} X \leqq n$.

Proof. That i) implies ii): Let X_i' be a replica of X_i. Set

$$P_i = \prod_{j<i} X_j \times \prod_{j \geqq i} X_j'.$$

Then P_i is σ-metric. Especially P_1 is metric. X is the inverse limit of $\{P_i\}$ with the natural projections $g^i{}_j$. Let P_{ik} be the product of the first k factors of P_i. Then P_{ik} is σ-homeomorphic onto the product of the first k factors of X, say P_{ik}', where a mapping onto is said σ-homeomorphic if the domain is the countable sum of closed sets each of which is mapped homeomorphically to a closed set of the range. Hence we have $\dim P_{ik} = \dim P_{ik}' \leqq \dim X \leqq n$. Since P_i is the inverse limit of $\{P_{ik} : k=1,2,\cdots\}$, we have $\dim P_i \leqq n$ by Lemma 2. Since P_1 is metric, there exist a metric space Q_1 with $\dim Q_1 \leqq 0$ and a perfect mapping f_1 of Q_1 onto P_1 with $\operatorname{ord} f_1 \leqq n+1$ (cf. [4, Theorem 12.6]). Look at the diagram:

$$\begin{array}{ccc} Q_i & \xrightarrow{f_i} & P_i \\ {\scriptstyle h^i{}_1}\downarrow & & \downarrow{\scriptstyle g^i{}_1} \\ Q_1 & \xrightarrow{f_1} & P_1 \end{array}$$

By Lemma 5 there exist, for each i, a paracompact σ-metric space Q_i, a perfect mapping f_i of Q_i onto P_i and a σ-homeomorphic mapping $h^i{}_1$ of Q_i onto Q_1 such that $f_1 h^i{}_1 = g^i{}_1 f_i$ and such that the topology of Q_i is the weakest one to enable f_i and $h^i{}_1$ to be continuous. For each pair $i>j$ define $h^i{}_j : Q_i \to Q_j$ in such a way that $h^i{}_j = (h^j{}_1)^{-1} h^i{}_1$. Then $f_j h^i{}_j = g^i{}_j f_i$. Let $\mathfrak{U}_1$ be the topology of Q_1 and $\mathfrak{B}_k$ the topology of P_k. Since

$$\begin{aligned} &(h^i{}_j)^{-1}((h^j{}_1)^{-1}(\mathfrak{U}_1) \wedge f_j{}^{-1}(\mathfrak{B}_j)) \\ &= (h^i{}_1)^{-1}(\mathfrak{U}_1) \wedge (h^i{}_j)^{-1} f_j{}^{-1}(\mathfrak{B}_j) \\ &= (h^i{}_1)^{-1}(\mathfrak{U}_1) \wedge f_i{}^{-1}(g^i{}_j)^{-1}(\mathfrak{B}_j) \\ &\subset (h^i{}_1)^{-1}(\mathfrak{U}_1) \wedge f_i{}^{-1}(\mathfrak{B}_i), \end{aligned}$$

then $h^i{}_j$ is continuous. Let Z be the inverse limit of $\{Q_i\}$ and $f : Z \to X$ a transformation defined by: $g_1 f = f_1 h_1$, where $g_i : X \to P_i$ and $h_i : Z \to Q_i$ are the projections. Then f is obviously continuous. Every point-inverse under f is compact, since it is homeomorphic to the corresponding point-inverse under f_1. To prove the closedness of f let F be a closed set of Z and p a point from $X - f(F)$. Since $f^{-1}(p) \cap F = \emptyset$ and

$f^{-1}(p)$ is compact, there exist a k and an open set U of Q_k such that $f^{-1}(p)\subset h_k^{-1}(U)\subset Z-F$. Set $V=P_k-f_k(Q_k-U)$. Then V is open by the closedness of f_k. Since $g_k^{-1}(V)$ is an open neighborhood of p and $g_k^{-1}(V)\cap f(F)=\emptyset$, p is not in the closure of $f(F)$, proving the closedness of f. Of course ord $f=\text{ord} f_1\leqq n+1$. Since $\dim Q_i=\dim Q_1\leqq 0$ for each i, $\dim Z\leqq 0$ by Lemm 2.

That ii) implies iii): Set $H_i=\{x\in X: |f^{-1}(x)|=i\}$. Then X is the sum of $\{H_i: i=1,\cdots,n+1\}$. Since $f^{-1}(H_i)$ and H_i are paracompact σ-spaces, $\dim H_i\leqq 0$ by Lemma 4.

That iii) implies iv) or iv) implies i) is well known to be true for merely hereditarily normal spaces or normal spaces, respectively (cf. [4]). The proof is finished.

References

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