53. On a Theorem of I. Glicksberg

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§1. Let A be a function algebra on a compact Hausdorff space X. Some time ago Hoffman and Wermer [4] showed that the set of real parts $\operatorname{Re} A$ of A cannot be closed in $C_R(X)$ unless $A=C(X)$. As a consequence of the Hoffman-Wermer result, Glicksberg [3] has recently proved the following theorem: Let A be a function algebra on a compact metric space X and I be a closed ideal in A. If $A+\bar{I}$ is a closed, then $\bar{I}=I$, where $\bar{I}$ denotes the conjugate of I, i.e., $\bar{I}=\{\bar{f}; f\in I\}$. The main purpose of this paper is to give some extensions of the Glicksberg theorem in the case where X is any compact Hausdorff space.

By a function algebra on X we denote a closed subalgebra in $C(X)$ containing constant functions and separating points in X, where $C(X)$ is the Banach algebra of all complex-valued continuous functions on X with the uniform norm. Throughout this paper X will indicate a compact Hausdorff space.

Our results are following

Theorem 1. *Let A be a function algebra on a compact Hausdorff space X. Let N be a closed linear subspace in $C(X)$ and I be a closed ideal in A with $A+\bar{I}\supset N\supset I$. If $N+\bar{I}$ is closed, then $\bar{I}=I$.*

Theorem 2. *Let A be a function algebra on X. Let N be a closed linear subspace in A, I be a closed ideal in A and $N\cap I$ be an ideal in A. If $N+\bar{I}$ is closed, then $\overline{N\cap I}=N\cap I$.*

Theorem 3. *Suppose A is a function algebra on X and I, J are any two closed ideals in A. Then $I+\bar{J}$ is closed if and only if $\overline{I\cap J}=I\cap J$.*

§2. The following lemma is basic in our forthcoming proofs of these theorems.

Lemma 1. *Let A be a function algebra on X. Let N be a closed linear subspace in $C(X)$ and I be a closed ideal in A. If $N+\bar{I}$ is closed, there is $c>0$ such that $c\|g+(N\cap\bar{I})\|\leq\|\operatorname{Re} g\|$ for any $g\in N\cap I$, where* $\operatorname{Re} g$ *denotes the real part of g and $\|f+(N\cap\bar{I})\|$ is the norm of the factor space $(N+\bar{I})/(N\cap\bar{I})$,* i.e., $\|f+(N\cap\bar{I})\|=\inf_{h\in N\cap\bar{I}}\|f+h\|$.

Proof. We note first that the mapping $\Phi: f+\bar{g}+(N\cap\bar{I})\to f+(N\cap\bar{I})$ $(f\in N, g\in I)$ is well-defined as a linear mapping from the factor space $(N+\bar{I})/(N\cap\bar{I})$ to $N/(N\cap\bar{I})$. For, if $(f_1+\bar{g}_1)-(f_2+\bar{g}_2)\in N\cap\bar{I}$

$(f_1, f_2 \in N, g_1, g_2 \in I)$, then $f_1 - f_2 \in \bar{I}$. On the other hand, $f_1 - f_2 \in N$ since $f_1, f_2 \in N$, and so $f_1 - f_2 \in N \cap \bar{I}$. Thus Φ is well-defined. Next we shall prove that Φ is a continuous linear mapping. For this it is enough to show that Φ is a closed linear mapping by the closed graph theorem, since $(N+\bar{I})/(N\cap\bar{I})$ and $N/(N\cap\bar{I})$ are both Banach spaces. Let $f_n + \bar{g}_n + (N\cap\bar{I})$ converge to $\boldsymbol{O}$ (=the zero element in $(N+\bar{I})/(N\cap\bar{I})$) in $(N+\bar{I})/(N\cap\bar{I})$ and $f_n + (N\cap\bar{I})$ tend to $p+(N\cap\bar{I})$ in $N/(N\cap\bar{I})$, where $f_n, p \in N$ and $g_n \in I$. Then $\bar{g}_n + (N\cap\bar{I})$ converges to $-p+(N\cap\bar{I})$ in $(N+\bar{I})/(N\cap\bar{I})$. Since $\bar{I}/(N\cap\bar{I})$ is closed in $(N+\bar{I})/(N\cap\bar{I})$ and $p\in N$, we have that $p \in N\cap\bar{I}$, and so $p+(N\cap\bar{I})=\boldsymbol{O}$ in $N/(N\cap\bar{I})$. We are done. Now, since Φ is continuous, we have for some constant $c>0$.

$$2c\|f+(N\cap\bar{I})\| \le \|f+\bar{g}+(N\cap\bar{I})\| \qquad (f\in N, g\in I).$$

If $g\in N\cap I$, put $f=g$ in the above inequality. Then we obtain the desired one: $c\|g+(N\cap\bar{I})\| \le \|\mathrm{Re}\, g\|$.

Lemma 2. *Suppose N is a linear subspace in $C(X)$. If* $\mathrm{Re}\, N$ *is closed, then so is* $\mathrm{Re}\,(N+\boldsymbol{C})$, *where* $\mathrm{Re}\, N$ *is the real part of N, i.e.,* $\mathrm{Re}\, N = \{\mathrm{Re}\, f\,;\, f\in N\}$ *and $\boldsymbol{C}$ denotes the space of all complex numbers.*

Proof. Let $\mathrm{Re}\, f_n + r_n \to h$ ($f_n \in N, r_n$: real number). Then we shall show that $h\in \mathrm{Re}\,(N+\boldsymbol{C})$. (i) If $r_{n_i}\to r$ for a subsequence $\{r_{n_i}\}$ of $\{r_n\}$, then $\mathrm{Re}\, f_{n_i} \to h-r$ since $\mathrm{Re}\, f_{n_i}+r\to h$. Since $\mathrm{Re}\, N$ is closed, $h-r\in \mathrm{Re}\, N$. Hence $h\in\mathrm{Re}\,(N+\boldsymbol{C})$. (ii) If $r_{n_i}\to\infty$ (or $-\infty$) for a subsequence $\{r_{n_i}\}$ of $\{r_n\}$, then $\mathrm{Re}\,(f_{n_i}/r_{n_i})+1\to 0$. Therefore $\mathrm{Re}\, N \ni 1$ since $\mathrm{Re}\, N$ is closed. If $\mathrm{Re}\, f_0 = 1$ ($f_0\in N$), then $\mathrm{Re}\,(f_n + r_n f_0)\to h$. This shows that $h\in \mathrm{Re}\, N \subset \mathrm{Re}\,(N+\boldsymbol{C})$, because $f_n + r_n f_0 \in N$ and $\mathrm{Re}\, N$ is closed. It completes the proof.

§3. With these preparation we shall give here proofs of our theorems.

Proof of Theorem 1. If we put $M=\{f\in A\,;\, f+\bar{g}\in N$ for some $g\in I\}$, then M is closed in A and $M+\bar{I}=N+\bar{I}$. In order to prove that M is closed, let $f_n\in M$ and $f_n\to f$. Then $f_n+\bar{g}_n\in N$ for some $g_n\in I$ $(n=1,2,3,\cdots)$. Since $f_n\in N+\bar{I}$ and $N+\bar{I}$ is closed, $f\in N+\bar{I}$. If we put $f=h-\bar{g}$ $(h\in N, g\in I)$, we have $f+\bar{g}=h\in N$, and so $f\in M$ by the definition of M. Hence, without loss of generality, we may assume that $A\supset N$. From Lemma 1, we have

$$(*) \qquad c\|g+(N\cap\bar{I})\| \le \|\mathrm{Re}\, g\| \qquad (g\in I).$$

This will imply that $\mathrm{Re}\, I$ is closed, so $\mathrm{Re}\,(I+\boldsymbol{C})$ is also closed (by Lemma 2). If it was shown, the proof of the theorem would be immediate. For, if $\mathrm{Re}\,(I+\boldsymbol{C})$ is closed, then $\overline{I+\boldsymbol{C}}=I+\boldsymbol{C}$ by the Hoffman-Wermer theorem (cf. [4], Glicksberg [3]). This follows that $\bar{I}=I$. For, if $\overline{I+\boldsymbol{C}}=I+\boldsymbol{C}$, then $I|K\equiv\boldsymbol{C}$ or 0 for any maximal antisymmetric set K for A. Hence, by a theorem of Glicksberg ([2], Theorem 2.5), we have $\bar{I}=I$. From this, in order to verify the theorem, it remains to show that $\operatorname{Re} I$ is closed. Let h be an arbitrary point in the uniform closure of $\operatorname{Re} I$. Then $\operatorname{Re} g_n$ converges to h for a sequence $\{g_n\}\subset I$. By the above inequality $(*)$, we see that $\{g_n+(N\cap \bar{I})\}$ is a Cauchy sequence in the Banach space $N/(N\cap \bar{I})$, and so there exist an $s\in N$ and some $s_n\in N\cap \bar{I}$ $(n=1,2,3,\cdots)$ such that

$$(**)\qquad \|g_n-s+s_n\|\to 0 \qquad (n\to\infty).$$

Here we remark that $N\cap \bar{I}\subset I$ (since $N\subset A$). The fact is an easy consequence of a result of Glicksberg ([2], Corollary 2.6). Hence by $(**)$, $s\in I$. From $(**)$ again, we have

$$\|\operatorname{Re} g_n-\operatorname{Re} s+\operatorname{Re} s_n\|\to 0 \qquad (n\to\infty).$$

Since $\bar{s}_n\in \overline{N\cap \bar{I}}\subset I$, $\operatorname{Re} s_n\in I$, and it implies that $h-\operatorname{Re} s\in I$ because $\operatorname{Re} g_n\to h$. If we put $q=h-\operatorname{Re} s$, then $q\in \operatorname{Re} I$ since q is real. Hence $h\in \operatorname{Re} I$ and so $\operatorname{Re} I$ is closed. The proof is complete.

Proof of Theorem 2. The proof is almost same as one of Theorem 1. By Lemma 1, we have for some $c>0$,

$$c\|g+(N\cap \bar{I})\|\le\|\operatorname{Re} g\| \qquad (g\in N\cap I).$$

Put $J=N\cap I$. Then J is an ideal by the hypothesis. If it was shown that $\operatorname{Re} J$ is closed, then the theorem would be proved as in the proof of Theorem 1. Therefore it suffices to show only that $\operatorname{Re} J$ is closed. For this, let $\operatorname{Re} g_n\to h$ $(g_n\in J)$. From the above inequality we see that $\{g_n+(N\cap \bar{I})\}$ is a Cauchy sequence in $N/(N\cap \bar{I})$. Hence, for an $s\in N$ and some $s_n\in N\cap \bar{I}$,

$$\|g_n-s+s_n\|\to 0 \qquad (n\to\infty).$$

Since $N\subset A$, $N\cap \bar{I}\subset N\cap I=J$, and hence $s\in J$. Also we have

$$\|\operatorname{Re} g_n-\operatorname{Re} s+\operatorname{Re} s_n\|\to 0 \qquad (n\to\infty).$$

But $\bar{s}_n\in \overline{N\cap \bar{I}}\subset \bar{J}\cap I\subset J$ (cf. [2], Corollary 2.6). It follows that $\operatorname{Re} s_n\in J$. Hence $h-\operatorname{Re} s\in J$. From this, it can be shown that $\operatorname{Re} J$ is closed as in the proof of Theorem 1.

Proof of Theorem 3. The necessity of the proof is clear from Theorem 2. Hence we have only to prove the sufficiency. Suppose $\overline{I\cap J}=I\cap J$. Then we easily see that $I\cap J=\{f\in C(X)\,;\,f(H)=0\}$, where $H=$the hull of $I\cap J=\bigcap_{f\in I\cap J} Z(f)$, $Z(f)=\{x\in X\,;\,f(x)=0\}$. If $H=\phi$, then the proof is immediate, so let $H\neq\phi$. Since I, J are ideals in A, we have $H=H_1\cup H_2$, where H_1 and H_2 denote the hulls of I and J respectively. In order to prove that $I+\bar{J}$ is closed, we suppose that $f_n+\bar{g}_n\to h$ $(f_n\in I, g_n\in J)$. Then $f_n\to h$ on H_2 and $f_n\to 0$ on H_1, because $f_n=0$ on H_1 $(n=1,2,3,\cdots)$. Hence $\{f_n|H\}$ is a Cauchy sequence, that is, for any $\varepsilon>0$, there is a positive integer N such that for $m,n\ge N$, $\|f_n-f_m\|_H<\varepsilon$. Choosing a suitable subsequence $\{f_{n_i}\}$ in $\{f_n\}$, we have $\|f_{n_{i+1}}-f_{n_i}\|_H<2^{-i}$ $(i=1,2,3,\cdots)$. Here we define sequences $\{p_i\}$, $\{h_i\}$ of continuous functions on X as follows: $p_1\equiv f_{n_1}$, $h_i=f_{n_{i+1}}-f_{n_i}$ on H, $\|h_i\|_X<2^{-i}$ and $p_{i+1}=p_i+h_i$ $(i=1,2,3,\cdots)$. Then $h_i-(f_{n_{i+1}}-f_{n_i})$

$\in I \cap J$, since $h_i-(f_{n_{i+1}}-f_{n_i})=0$ on H. Hence $h_i \in I$ and so $p_i \in I$ ($i=1, 2, 3, \cdots$). Also, since $\|p_{i+1}-p_i\|=\|h_i\|<2^{-i}$, p_i converges uniformly to a function $p \in I$. We see that $p_i=f_{n_i}$ on H, and $q_i-g_{n_i}=\bar{f}_{n_i}-\bar{p}_i=0$ on H if $q_i=g_{n_i}+\bar{f}_{n_i}-\bar{p}_i$ ($i=1, 2, 3, \cdots$). Hence $q_i-g_{n_i} \in I \cap J$, and so $q_i \in J$ ($i=1, 2, 3, \cdots$). Since $q_i=g_{n_i}+\bar{f}_{n_i}-\bar{p}_i$, $g_{n_i}+\bar{f}_{n_i}\to\bar{h}$ and $p_i\to p$, we have $q_i\to\bar{h}-\bar{p}$. If we put $q=\bar{h}-\bar{p}$, then $q \in J$, and so $p+\bar{q}=h$ ($p \in I, q \in J$). The theorem is proved.

Remark. Under the assumption of Theorem 1, the following conditions are equivalent: (1) $N+\bar{I}$ is closed. (2) $\bar{I}=I$. (3) $X\sim hI$ is a w-interpolation set for A, that is, any compact subset in $X\sim hI$ is an interpolation set (cf. [5] or [6]), where hI is the hull of I. The equivalence of (2) and (3) can be proved as follows: If (2) holds, then $hI\supset E$ (cf. [3]), where E denotes the essential set for A ([1]). Hence $X\sim hI \subset X\sim E$, and so $X\sim hI$ is a w-interpolation set. Conversely, let $X\sim hI$ be a w-interpolation set. Then $X\sim hI\subset X\sim\partial_{A|E}$ (cf. [5] or [6]). Therefore $hI\supset\partial_{A|E}$ and it follows that $hI\supset E$. From this we can easily see that $\bar{I}=I$.

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