51. A Note on the Dilation Theorems

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1. Introduction. In the recent decade, the so-called harmonic analysis of operators grew rapidly by the works mainly due to Sz. Nagy's school, cf. [5]. The main tool in their investigations is the following strong dilation theorem due to Sz. Nagy:

Theorem A. *If T is a contraction acting on a Hilbert space $\mathfrak{H}$, then there is a unitary U acting on a Hilbert space $\mathfrak{K}$ including $\mathfrak{H}$ as a subspace such that*

(1) $$T^n = PU^n | \mathfrak{H} \qquad (n=0, 1, 2, \cdots),$$

where P is the projection of $\mathfrak{K}$ onto $\mathfrak{H}$.

By the importance of the theorem, several proofs are given, cf. [5; Chapter I]. Some of them are based on the following general dilation theorems due to Naimark, cf. [3], [5].

Theorem B. *If $F(A)$ is a positive operator-valued measure defined on a σ-field $\mathfrak{B}$ of sets and $F(A)$ acts on $\mathfrak{H}$, then there is a spectral measure $E(A)$ of $\mathfrak{B}$ acting on $\mathfrak{K}$ including $\mathfrak{H}$ such that*

(2) $$F(A) = PE(A) | \mathfrak{H} \qquad (A \in \mathfrak{B}).$$

Theorem C. *If $V(g)$ is an operator-valued positive definite function defined on a group G and $V(g)$ acts on $\mathfrak{H}$, then there is a unitary representation $U(g)$ of G on $\mathfrak{K}$ including $\mathfrak{H}$ such that*

(3) $$V(g) = PU(g) | \mathfrak{H} \qquad (g \in G).$$

However, there is an another general dilation theorem due to Stinespring [4] and Umegaki [6] which receives less attentions:

Theorem D. *If $V(a)$ is a completely positive (or positive definite in the sense of [6]) linear mapping of a $*$-algebra $\mathcal{A}$ into $\mathcal{B}(\mathfrak{H})$, the algebra of all (bounded linear) operators acting on $\mathfrak{H}$, then there is a $*$-homomorphism $\Phi(a)$ of $\mathcal{A}$ into $\mathcal{B}(\mathfrak{K})$ where $\mathfrak{K}$ includes $\mathfrak{H}$ and Φ satisfies*

(4) $$V(a) = P\Phi(a) | \mathfrak{H} \qquad (a \in \mathcal{A}).$$

It seems to the authors that there is no literature which gives a proof that Theorem D implies Theorem A. In § 2, we shall give some theorems proofs.

Umegaki [6] pointed out that Theorem C implies Theorem D if $\mathcal{A}$ is the group algebra of a locally compact group G. The converse of this implication obviously follows from

$$(5) \qquad V(a)=\int_G a(g)V(g)dg \qquad (a \in L^1(G)).$$

Hence, Theorems C and D are equivalent if G is locally compact.

In § 3, we shall show that Theorems B and D are equivalent if $\mathcal{A}$ is abelian by the help of the following theorem due to Stinespring [4]:

Theorem E. *If $\mathcal{A}$ is abelian, then the complete positivity of V coincides with the usual positivity.*

2. Implication. Here we shall show

Theorem 1. *Theorem D implies Theorem A.*

Let $\mathcal{A}$ be the algebra of all (complex valued) functions on $[0, 2\pi]$ with absolutely summable Fourier coefficients; the multiplication of $\mathcal{A}$ is the convolution, $*$-operation is given by

$$f^*(\theta)=\sum_{n=-\infty}^{\infty} \alpha_{-n}{}^* e^{in\theta}$$

and the norm is given by

$$\|f\|=\sum_{n=-\infty}^{\infty} |\alpha_n|,$$

for

$$f(\theta)=\sum_{n=-\infty}^{\infty} \alpha_n e^{in\theta}.$$

Obviously $\mathcal{A}$ is isometrically isomorphic to the group algebra $l^1(Z)$ of the group Z of all integers.

Let T be a contraction on $\mathfrak{H}$. Then we can define a linear map V of $\mathcal{A}$ into $\mathcal{B}(\mathfrak{H})$ by

$$(6) \qquad V(f)=\sum_{n=-\infty}^{\infty} \alpha_n T^{(n)} \qquad \text{for } f=\sum_{n=-\infty}^{\infty} \alpha_n e^{in\theta} \in \mathcal{A},$$

where

$$(7) \qquad T^{(n)}=\begin{cases} T^n & (n>0) \\ I & (n=0) \\ T^{*|n|} & (n<0). \end{cases}$$

For any positive element $f \in \mathcal{A}$ and $x \in \mathfrak{H}$, we have

$$(V(f)x \mid x)=\sum_{n=-\infty}^{\infty} \alpha_n (T^{(n)}x \mid x)=\alpha_0\|x\|^2+2\,\mathrm{Re}\sum_{n=1}^{\infty} \alpha_n (T^n x \mid x).$$

By the theorem of Herglotz-Bochner, we have

$$\alpha_n=\int_0^{2\pi} e^{in\theta} d\mu(\theta) \qquad (n \geqq 1),$$

so that we have

$$(8) \qquad (V(f)x \mid x)=\|x\|^2+2\,\mathrm{Re}\sum_{n=1}^{\infty} (T^n x \mid x)\int_0^{2\pi} e^{in\theta} d\mu(\theta).$$

Now, we shall employ at technique due to Foias [2]: For every complex number z with $0 \leqq |z| < 1$, we have

$$\mathrm{Re}\,[I+2\sum_{n=1}^{\infty} (zT)^n]=\mathrm{Re}\,(I+zT)(I-zT)^{-1} \geqq 0,$$

so that we have

$$\|x\|^2+2\,\mathrm{Re}\sum_{n=1}^{\infty}(T^n x\,|\,x)r^n e^{in\theta}\geqq 0,$$

for $0\leqq r<1$ and $0\leqq\theta\leqq 2\pi$. Hence, integrating, we have

$$\|x\|^2+2\,\mathrm{Re}\sum_{n=1}^{\infty}(T^n x\,|\,x)r^n\int_0^{2\pi}e^{in\theta}d\mu(\theta)\geqq 0,$$

for every $0\leqq r<1$. Tending r to 1, we have $V(f)\geqq 0$ by (8).

We shall now utilize Theorem E. Although Theorem E is proved for C^*-algebras, we can modify the proof of Stinespring in our present setting. Hence we can conclude that V is positive definite.

Applying Theorem D, we have a $*$-homomorphism Φ of $\mathcal{A}$ into $\mathcal{B}(\mathfrak{K})$ such that $V(f)=P\Phi(f)\,|\,\mathfrak{H}$ for any $f\in\mathcal{A}$. Let us put

$$U=\Phi(e^{i\theta}).$$

U is clearly a unitary operator acting on $\mathfrak{K}$, and we have (1).

Our second proof is similar to that of Sz. Nagy-Foiaş [5; p. 27f]. Let $\mathcal{A}$ and V be as in the above. Let $\mathcal{A}_0$ be the set of all functions in $\mathcal{A}$ whose coefficients vanish up to finite numbers. $\mathcal{A}_0$ is a $*$-subalgebra of $\mathcal{A}$. We shall try to prove directly the complete positiveness of V. For $x_1,\cdots,x_n$ and $f_1,\cdots,f_n$, where

$$f_k(\theta)=\sum_{j=-\infty}^{\infty}\alpha_j^{(k)}e^{ij\theta}\in\mathcal{A}_0,$$

we have

$$\begin{aligned}D&=\sum_{k,m=1}^{n}(V(f_m^* * f_k)x_k\,|\,x_m)\\&=\sum_{k,m=1}^{n}\sum_{s,t=-\infty}^{\infty}\alpha_s^{(m)*}\alpha_t^{(k)}(T^{(t-s)}x_k\,|\,x_m)\\&=\sum_{s,t=-\infty}^{\infty}(T^{(t-s)}y_t\,|\,y_s)\end{aligned}$$

where

$$y_t=\sum_{k=1}^{n}\alpha_t^{(k)}x_k.$$

(Replacing y_t by y_{t+c} if necessary, we may assume that $y_t=0$ for $t<0$). If we put

$$z_t=\sum_{t\leqq s}T^{s-t}y_s,$$

then we have

$$\begin{aligned}D&=\sum_{s,t\geqq 0}(T^{(t-s)}(z_t-Tz_{t+1})\,|\,(z_s-Tz_{s+1}))\\&=\sum_{s,t\geqq 0}(D(t,s)z_t\,|\,z_s),\end{aligned}$$

where

$$D(t,s)=\begin{cases}I-T^*T & (t=s\geqq 1)\\ I & (t=s=0)\\ 0 & (\text{otherwise}).\end{cases}$$

Therefore we have

$$D=\|z_0\|^2+\sum_{t\geqq 1}((I-T^*T)z_t\,|\,z_t)\geqq 0,$$

which proves that V is positive definite. The remainder of the proof as same as that of the first proof.

Our third proof is a variant of the second and essentially due to Foiaş [2]. In the second proof, we can change into the following:

$$\begin{aligned} D &= \sum_{k,m=1}^{n} (V(f_m^* * f_k)x_k \mid x_m) \\ &= \sum_{s,t=0}^{\infty} (T^{(t-s)}y_t \mid y_s) \\ &= \lim_{0<r\uparrow 1} \sum_{s,t=0}^{\infty} r^{|t-s|}(T^{(t-s)}y_t \mid y_s) \\ &= \lim_{0<r\uparrow 1} \frac{1}{2\pi}\int_0^{2\pi} p(r;\theta)d\theta \geqq 0, \end{aligned}$$

where

$$p(r;\theta) = \mathrm{Re}\left((I+re^{i\theta})(I-re^{i\theta})^{-1}\sum_{t=-\infty}^{\infty} e^{-it\theta}y_t \,\Big|\, \sum_{s=-\infty}^{\infty} e^{is\theta}y_s\right).$$

3. Equivalence. In this section, we shall show

Theorem 2. *Theorems B and D are equivalent for abelian C^*-algebras.*

Let us assume Theorem B. If $\mathcal{A}$ is an abelian C^*-algebra and V is a positive definite mapping of $\mathcal{A}$ into $\mathcal{B}(\mathfrak{H})$. Then, by the Gelfand representation theorem, $\mathcal{A}$ is $*$-isomorphic to the algebra $C(X)$ of all continuous functions on a compact Hausdorff space X. Let $\mathfrak{B}$ be the σ-field of Borel subsets of X. Then if we put $F_{\xi,\eta} = (V(f)\xi \mid \eta)$ we obtain a semi-spectral measure $F(A)$ of $\mathfrak{B}$ on $\mathfrak{H}$ such that

$$F_{\xi,\eta}(f) = \int_X f(x)d(F(x)\xi \mid \eta)$$

for $f \in \mathcal{A}$ and $\xi, \eta \in \mathfrak{H}$. Therefore by the hypothesis there exists a spectral measure $E(A)$ of $\mathfrak{B}$ on $\mathfrak{K} \supset \mathfrak{H}$ such that they satisfy (2).

Let us now define a linear map Φ of $\mathcal{A}$ into $\mathcal{B}(\mathfrak{K})$ by

$$\Phi(f) = \int_X f(x)dE(x) \qquad (f \in \mathcal{A}), \tag{9}$$

then Φ is a $*$-homomorphism of $\mathcal{A}$ on $\mathfrak{K}$ since $E(A)$ is a spectral measure, and we have (3) as desired.

Conversely, let $\mathfrak{B}$ be a σ-field of sets and $F(A)$ be a semi-spectral measure of $\mathfrak{B}$ on $\mathfrak{H}$. Let $\mathcal{A}$ be an abelian C^*-algebra generated by the characteristic functions of sets of $\mathfrak{B}$ with the sup-norm. If we define a linear map V of $\mathcal{A}$ into $\mathcal{B}(\mathfrak{H})$ by

$$V(f) = \int_X f(x)dF(x) \qquad (f \in \mathcal{A}), \tag{10}$$

then V is positive definite since $F(A)$ is a semi-spectral measure. Therefore, by Theorem D, there is a $*$-homomorphism Φ of $\mathcal{A}$ into $\mathcal{B}(\mathfrak{K})$ where $\mathfrak{K} \supset \mathfrak{H}$ such that they satisfy (4).

By the spectral theorem, there is a spectral measure E(A) which satisfies (9). Putting $f=\chi_A$, (4) implies (2), where χ_A is the characteristic function of $A \in \mathfrak{B}$.

4. Remark. In the final stage of the preparation of the present note, the authors are awared by Prof. H. Choda that a similar task for § 2 is announced in a paper of Arveson [1]. We suppose that our proof is somewhat different by the use of $l^1(Z)$ which is a $*$-algebra but not a C^*-algebra.

References

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