50. On Normal Approximate Spectrum

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1. Introduction. Bunce [1] established a kind of the reciprocity among the characters of singly generated C^*-algebras and the approximate spectra of the generators of certain classes. Kasahara and Takai [5] introduced the notion of the normal approximate spectra and gave new proofs of the main theorems of Bunce [1]. However, Kasahara and Takai remain the reciprocity of characters and spectra.

In the present note, we shall complete the reciprocity of Bunce with the use of the idea due to Kasahara and Takai in § 2. We shall also discuss the joint normal approximate spectra in § 3. In § 4, we shall show that a theorem of Coburn [3] is given an elementary proof based on the reciprocity obtained in § 2.

2. Reciprocity. Let A be a (bounded linear) operator on a Hilbert space $\mathfrak{H}$. Let $\pi(A)$ be the approximate spectrum of A, cf. [4]. Following after the definition of Kasahara and Takai [5], a complex number λ is a *normal approximate propervalue* of A if there exists a sequence $\{x_n\}$ of unit vectors such that

$$\|(A-\lambda)x_n\|\to 0 \quad \text{and} \quad \|(A-\lambda)^*x_n\|\to 0 \quad (n\to\infty).$$

The set $\pi_n(A)$ of all normal approximate propervalues is called the *normal approximate spectrum* of A.

Let us begin to prove the following lemma which is essentially due to Mr. H. Takai:

Lemma 1. *$\lambda\in\pi_n(A)$ if and only if $(A-\lambda)^*(A-\lambda)+(A-\lambda)(A-\lambda)^*$ is not strictly positive,* i.e. *there is no $\varepsilon>0$ such that*

$$(A-\lambda)^*(A-\lambda)+(A-\lambda)(A-\lambda)^*\geqq\varepsilon. \tag{1}$$

Proof. By the equality

$$(((A-\lambda)^*(A-\lambda)+(A-\lambda)(A-\lambda)^*)x\,|\,x) = \|(A-\lambda)x\|^2+\|(A-\lambda)^*x\|^2, \tag{2}$$

$\lambda\in\pi_n(A)$ implies there is a sequence $\{x_n\}$ of unit vectors such that the both terms of the right-hand side of (2) tend to 0, so that (1) is not satisfied.

Conversely, if (1) is not satisfied, then 0 is contained in the closure $\bar{W}((A-\lambda)^*(A-\lambda)+(A-\lambda)(A-\lambda)^*)$ of the numerical range of the operator described in the left-hand side of (1), so that $\lambda\in\pi_n(A)$.

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By the above observation, we have an another equivalent formulation for the definition of an normal approximate propervalue of A; that is, $\lambda \in \pi_n(A)$ *if and only if*

$$0 \in \bar{W}((A-\lambda)^*(A-\lambda)+(A-\lambda)(A-\lambda)^*). \tag{3}$$

Let $\mathfrak{A}$ be the C^*-algebra generated by A and the identity I. By a *character* of $\mathfrak{A}$ we mean a multiplicative linear functional of $\mathfrak{A}$. We shall show here the following reciprocity among the characters of $\mathfrak{A}$ and the normal approximate spectrum of A:

Theorem 1. *The following conditions are equivalent:*

(i) $\lambda \in \pi_n(A)$, *and*

(ii) *There is a character ϕ of $\mathfrak{A}$ such that*

$$\phi(A)=\lambda. \tag{4}$$

Proof. Kasahara and Takai [5] proved that (i) implies (ii). Hence we shall prove the converse. Suppose that (ii) is satisfied and λ is not a normal approximate propervalue of A. Then (1) is satisfied for some $\varepsilon>0$. Hence we have

$$\begin{aligned}0&=\phi(A-\lambda)^*\phi(A-\lambda)+\phi(A-\lambda)\phi(A-\lambda)^*\\&=\phi(A-\lambda)^*(A-\lambda)+(A-\lambda)(A-\lambda)^*)\geqq\varepsilon>0,\end{aligned}$$

which is a contradiction.

If A is hyponormal, then $\pi(A)=\pi_n(A)$ as proved in [5]. Hence Theorem 1 implies

Theorem 2 (Bunce). *If A is hyponormal, then $\lambda \in \pi(A)$ if and only if there is a character ϕ of $\mathfrak{A}$ which satisfies* (4).

3. Joint normal approximate spectrum. Let $A_1, \cdots, A_n$ be operators on $\mathfrak{H}$. According to [2] and [6], we say that a set of complex numbers $\lambda_1, \cdots, \lambda_n$ is a *joint approximate propervalue* of $A_1, \cdots, A_n$ if there is no $\varepsilon>0$ such that

$$(A_1-\lambda_1)^*(A_1-\lambda_1)+\cdots+(A_n-\lambda_n)^*(A_n-\lambda_n)\geqq\varepsilon. \tag{5}$$

The set $\pi(A_1, \cdots, A_n)$ of all joint approximate propervalues is called the *joint approximate spectrum* of $A_1, \cdots, A_n$. Nakamoto and Nakamura [6] proved that $(\lambda_1, \cdots, \lambda_n) \in \pi(A_1, \cdots, A_n)$ if and only if

$$\mathfrak{B}(\mathfrak{H})(A_1-\lambda_1)+\cdots+\mathfrak{B}(\mathfrak{H})(A_n-\lambda_n)\neq\mathfrak{B}(\mathfrak{H}), \tag{6}$$

where $\mathfrak{B}(\mathfrak{H})$ is the algebra of all operators on $\mathfrak{H}$. Bunce [2] proved that $\pi(A_1, \cdots, A_n)$ is not empty if $A_1, \cdots, A_n$ commute each other. Bunce [2] defined the joint approximate spectrum for commutative families of operators, whereas we do not claim the commutativity in our definition. In our case, the joint approximate spectrum is possibly void. The following theorem is an analogue of [6]:

Theorem 3. *The following conditions are equivalent:*

(i) *There are $\|x_k\|=1$ such that $\|A_i x_k-\lambda_i x_k\|\to 0$ for $i=1,2,\cdots,n$.*

(ii) *There are projections P_k such that $\|(A_i-\lambda_i)P_k\|\to 0$ for $i=1,2,\cdots,n$.*

(iii) $\mathfrak{A}(A_1-\lambda_1)+\cdots+\mathfrak{A}(A_n-\lambda_n)\neq\mathfrak{A}$ *where* $\mathfrak{A}$ *is the* C^**-algebra generated by* $A_1,\cdots,A_n$ *and* I.

(iv) *There is no* $\varepsilon>0$ *satisfying* (5).

Proof. (i) implies (ii): Put $P_k=x_k\otimes x_k$, where $(x\otimes y)z=(z\,|\,y)x$. Then we have (ii).

(ii) implies (iii): Let $\mathfrak{J}=\{B\in\mathfrak{A}\,;\,\|BP_k\|\to 0\}$. Then $\mathfrak{J}$ is a proper left ideal of $\mathfrak{A}$ including $\mathfrak{A}(A_1-\lambda_1)+\cdots+\mathfrak{A}(A_n-\lambda_n)$.

(iii) implies (iv): If (5) is satisfied by $\varepsilon>0$, then the left-hand side of (5) is invertible. Hence $\mathfrak{A}(A_1-\lambda_1)+\cdots+\mathfrak{A}(A_n-\lambda_n)$ contains I and contradicts to (iii).

(iv) implies (i): 0 is included in the closure of the numerical range of the left-hand side of (5). By (3), we have (i).

Kasahara and Takai [5] introduced the notion of normal approximate propervalues. Correspondingly, we shall introduce the following

Definition. $(\lambda_1,\cdots,\lambda_n)$ is a *joint normal approximate propervalue* of $A_1,\cdots,A_n$ if there is a sequence $\{x_k\}$ of unit vectors such that

(7) $$\|(A_i-\lambda_i)x_k\|\to 0 \quad\text{and}\quad \|(A_i-\lambda_i)^*x_k\|\to 0 \quad (k\to\infty)$$

for $i=1,2,\cdots,n$. The set $\pi_n(A_1,\cdots,A_n)$ of all joint normal approximate propervalues is called the *joint normal approximate spectrum* of $A_1,\cdots,A_n$.

From the definition and Lemma 1, we have

Corollary 1. $\lambda\in\pi_n(A)$ *if and only if* $(\lambda,\lambda^*)\in\pi(A,A^*)$.

From the definition and Theorem 3, we have

Corollary 2. $(\lambda_1,\cdots,\lambda_n)$ *belongs to* $\pi_n(A_1,\cdots,A_n)$ *if and only if* $(\lambda_1,\cdots,\lambda_n,\lambda_1^*,\cdots,\lambda_n^*)$ *belongs to* $\pi(A_1,\cdots,A_n,A_1^*,\cdots,A_n^*)$.

Corresponding to Theorem 3, we have

Theorem 4. *The following conditions are equivalent*:

(i) $(\lambda_1,\cdots,\lambda_n)\in\pi_n(A_1,\cdots,A_n)$,

(ii) *There are projections* P_k *such that*

(8) $$\|(A_i-\lambda_i)P_k\|\to 0 \quad\text{and}\quad \|(A_i-\lambda_i)^*P_k\|\to 0 \quad (k\to\infty)$$

for $i=1,2,\cdots,n$.

(iii) $A_1-\lambda_1,\cdots,A_n-\lambda_n$ *generate a proper ideal* $\mathfrak{J}$ *of* $\mathfrak{A}$.

(iv) *There is no* $\varepsilon>0$ *such that*

(9) $$\sum_{i=1}^{n}[(A_i-\lambda_i)^*(A_i-\lambda_i)+(A_i-\lambda_i)(A_i-\lambda_i)^*]\geqq\varepsilon.$$

Proof. By Theorem 3 and Corollary 2, we need to show that (ii) implies (iii) implies (iv). If (ii) is true and $I\in\mathfrak{J}$ then $\|P_k\|=\|IP_k\|\to 0$. If (iii) is true and (9) is satisfied, then the left-hand side of (9) is invertible and $\mathfrak{J}$ contains I.

Corresponding to Theorem 1, we can state the following

Theorem 5. *For an abelian family of operators* $A_1,\cdots,A_n$, $(\lambda_1,\cdots,\lambda_n)\in\pi_n(A_1\cdots,A_n)$ *if and only if there is a character* ϕ *on* $\mathfrak{A}$ *such that*

(10) $$\phi(A_i)=\lambda_i \qquad (i=1,2,\cdots,n).$$

Since the proof of Theorem 5 is completely analogous to that of Kasahara and Takai [5], we shall omit the details.

Theorem 6 (Bunce). *If $A_1,\cdots,A_n$ are hyponormal and commute each other, then $(\lambda_1,\cdots,\lambda_n)\in\pi(A_1,\cdots,A_n)$ if and only if there is a character ϕ on $\mathfrak{A}$ satisfying* (10).

This is a consequence of a theorem of Kasahara and Takai [5] and Theorem 5.

4. Coburn's theorem. Using Theorem 1, we shall show here the following theorem:

Theorem 7 (Coburn). *If $\mathfrak{A}$ is the C^*-algebra generated by the unilateral shift S of multiplicity* 1 *and the identity I, then $\mathfrak{A}$ contains $\mathfrak{C}=\mathfrak{C}(l^2)$, the algebra of all compact operators on l^2. $\mathfrak{A}/\mathfrak{C}$ is isometrically isomorphic to the algebra $C(T^1)$ of all continuous functions defined on the circumference T^1 of the unit disc.*

Proof. Since the unilateral shift S is hyponormal, we have

$$\pi_n(S)=\pi(S)=T^1,$$

by [4] and [5]. Hence, to prove the theorem, we need to show that *the kernel $\mathfrak{N}$ of all characters of $\mathfrak{A}$ coincides with $\mathfrak{C}$.*

At first, we show that $\mathfrak{A}$ *contains* $\mathfrak{C}$ *entirely.* For any $\varepsilon>0$ and $y\in\mathfrak{H}$, there is a polynomial p such that

$$||p(S)e_0-y||<\varepsilon \qquad \text{where } e_0=(1,0,0,\cdots).$$

Hence we have

$$\begin{aligned}||p(S)(e_0\otimes e_0)-y\otimes e_0|| &= ||(p(S)e_0\otimes e_0)-y\otimes e_0||\\ &= ||(p(S)e_0-y)\otimes e_0|| = ||p(S)e_0-y||<\varepsilon.\end{aligned}$$

Since $e_0\otimes e_0=I-SS^*\in\mathfrak{A}\cap\mathfrak{C}$, we have $y\otimes e_0\in\mathfrak{A}$. Therefore, we have $e_0\otimes y=(y\otimes e_0)^*\in\mathfrak{A}$ and $z\otimes y=(z\otimes e_0)(e_0\otimes y)\in\mathfrak{A}$ for every $y,z\in\mathfrak{H}$. Since $\mathfrak{C}$ is generated by dyads, $\mathfrak{A}$ contains $\mathfrak{C}$ entirely.

Now, let ϕ be a character of $\mathfrak{A}$. We shall show that *ϕ is not a vector state.* Suppose that $\phi(A)=(Ae\,|\,e)$ for some $||e||=1$. Since $\pi(S)$ T^1 and S^ne converges to 0 weakly, we have

$$1=|\phi(S)|=|\phi(S^n)|=|(S^ne\,|\,e)|\to 0 \qquad (n\to\infty)$$

and a contradiction.

Considering a character ϕ on $\mathfrak{A}$ as the restriction of a pure state on $\mathfrak{B}(l^2)$, we can deduce that ϕ vanishes on $\mathfrak{C}$. Since a pure state on $\mathfrak{B}(l^2)$ is either normal or singular and we have just proved that ϕ is not normal.

Now, we shall show finally that $\mathfrak{C}=\mathfrak{N}$. Since S^*S is congruent to SS^* modulo $\mathfrak{C}$, we set that the quotient algebra $\mathfrak{A}/\mathfrak{C}$ is abelian. Through the Gelfand representation, we can identify $\mathfrak{A}/\mathfrak{C}$ with the algebra of all continuous functions on a compact Hausdorff space X. In this algebra, $\mathfrak{N}/\mathfrak{C}$ coincides with the ideal of all functions vanishing on a non-trivial closed set F. If ϕ' is a character of $\mathfrak{A}/\mathfrak{C}$ which does not belong to F, then $\phi=\phi'\circ\Phi$ is a character of $\mathfrak{A}$ where Φ is the natural homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}/\mathfrak{C}$. Clearly, the kernel of ϕ can not contain $\mathfrak{N}$, which is a contradiction to the definition of $\mathfrak{N}$. Hence F is trivial and $\mathfrak{C}=\mathfrak{N}$.

From the fact that $\mathfrak{A}$ contains $\mathfrak{C}$, we have the following corollaries which are already pointed out by Coburn [3]:

Corollary 3. *S is irreducible in the sense that there is no proper reducing subspace.*

Corollary 4. *$\mathfrak{A}$ is strongly dense in $\mathfrak{B}(l^2)$.*

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