66. *On the Uniqueness of the Shortest Single Axiom for the Implicational Calculus of Propositions*

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By the ICP (implicational calculus of propositions), we mean the implicational fragment of the classical propositional calculus. It can be axiomatized in various ways by axiom schemes and the rules of substitution and detachment as inference rules. In 1948, Jan Łukasiewicz [1] showed that the single axiom

(1) $CCCpqrCCrpCsp$

suffices to characterize the ICP, and gave a proof sketch of the fact that this is one of the shortest single axioms that can characterize the ICP. He left open, however, the question whether (1) is the unique axiom of the shortest length. In 1968, Richard Tursman [2] showed that (1) is the only 13 letter single axiom with a possible exception of

(2) $CCpqCCCqrpCsq$.

We have checked the correctness of these results using a computer, and could verify furthermore that (2) does not function as a single axiom for the ICP. Thus we came to the conclusion that (1) is the unique shortest single axiom for the ICP.

For the purpose of checking formulas, we have used models, some of which have more than one designated values. If a formula A is valid in a model but B is not so, then B can not be derived from A. Hence, in order to eliminate a candidate axiom A, we have only to show the existence of a model in which A holds but not (1). For constructing models, we used a computer. Since the implication is the only logical connective in the ICP, a model can be expressed by a matrix giving the truth table of implication, in which the designated values will be shown with the asterisk.

Tursman claimed to have eliminated using three matrices I, II and III shown below, all the 13 letter formulas from single axiom candidates except the following eleven formulas:

$CCCpCqrqCCqss$	$CCCpqpCCprCsr$
$CCCpCqrsCCsqq$	$CCCpqrCCrpCsp$
$CCpqCCCrCpspq$	$CCCpqrCsCCrpp$
$CCpqCCCrCqspq$	$CCpqCrCCCpspq$
$CCpqCCCprpCsq$	$CCpqCrCCCqspq$.
$CCpqCCCqrpCsq$	

I

C	1	2	3	
*1	1	2	3	$p=2$
2	1	1	3	$q=3$
3	1	1	1	$r=1$
				$s=2$

II

C	1	2	3	4	
*1	1	2	3	3	$p=4$
2	1	1	3	3	$q=1$
3	1	2	1	1	$r=1$
4	1	1	1	1	$s=2$

III

C	1	2	3	4	
*1	1	2	3	3	$p=3$
2	1	1	3	3	$q=1$
3	1	2	1	1	$r=4$
4	1	2	2	1	$s=3$

We noticed, however, that also the formula

$$CCCpqpCrCCpss$$

can not be eliminated by these three matrices. In fact, if one assigns the values for variables as indicated to the right of each of these matrices, one sees that this formula gets an undesignated value. Tursman gave three other matrices IV, V and VI shown below to eliminate the above eleven formulas except (1) and (2). The above formula, overlooked by Tursman, happens to be also eliminated by those three. Thus we could confirm the main result of Tursman.

IV

C	1	2	3	4	5
*1	1	2	4	4	4
*2	1	1	4	4	4
3	1	2	1	2	2
4	1	1	1	1	1
5	1	2	3	3	3

V

C	1	2	3	4	5
*1	1	1	1	4	4
*2	1	1	4	4	4
*3	1	1	1	4	4
4	1	1	1	1	1
5	1	1	1	1	1

VI

C	1	2	3
*1	1	2	3
2	1	1	2
3	1	1	1

Our only task for eliminating (2) from the single axiom candidates is to find out some model satisfying (2) but not (1). In order to find matrices which have this property, we have written a computer program scanning exhaustively all the three by three matrices. Thus we have found the matrix

C	1	2	3
*1	1	3	2
2	1	3	1
3	1	1	2

to be satisfying (2) but not (1). In fact, when we put $p=2$ and $q=r=s=1$, (1) assumes the undesignated value 3 as follows:

$$CCC211CC12C12=CC11C33=C12=3.$$

Thus we conclude that (1) *is the unique shortest single axiom for the* ICP.

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References

[1] J. Łukasiewicz: The shortest axiom of the implicational calculus of propositions. Proceedings of the Royal Irish Academy, **52**, sec. A, 25–33 (1948).

[2] R. Tursman: The shortest axiom of the implicational calculus. Notre Dame J. Formal Logic, **9**, 351–358 (1968).