89. On Normal Approximate Spectrum. III

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1. Introduction. In the previous notes [3] and [5], we have discussed certain properties of the normal approximate spectra of operators on a Hilbert space $\mathfrak{H}$. A complex number λ is an *approximate propervalue* of T acting on $\mathfrak{H}$ if there is a sequence $\{x_n\}$ of unit vectors such that

$$(*) \qquad \|(T-\lambda)x_n\| \to 0 \qquad (n\to\infty).$$

The set $\pi(T)$ of all approximate propervalues is called the *approximate spectrum* of T. If there exists a sequence $\{x_n\}$ of unit vectors satisfying $(*)$ and

$$(**) \qquad \|(T-\lambda)^* x_n\| \to 0 \qquad (n\to\infty),$$

then λ is called a *normal approximate propervalue* of T, which is occasionally discussed by Hildebrandt [7], Stampfli [11] and Yoshino [12]. The set $\pi_n(T)$ of all normal approximate propervalues of T is called the *normal approximate spectrum* of T. In general, $\pi_n(T)$ is a compact set in the plane and possibly void. Several equivalent conditions are discussed in [3], [5] and [9].

In the present note, we shall discuss some additional properties of the normal approximate spectra of operators. In § 2, we shall give a characterization of convexoids in terms of the normal approximate spectra. In a certain sense, a convexoid has sufficiently many normal approximate propervalues (Theorem 1), which is suggested by Prof. Z. Takeda, to whom the authors express their heaty thanks. In § 3, the normal approximate spectrum of the tensor product of operators is observed.

2. A characterization of convexoids. An operator T acting on a Hilbert space $\mathfrak{H}$ is called a *convexoid* if

$$(1) \qquad \bar{W}(T) = \mathrm{co}\,\sigma(T),$$

where $\bar{W}(T)$ is the closure of the numerical range $W(T)$ given by

$$(2) \qquad W(T) = \{(Tx \mid x)\,;\, \|x\|=1\},$$

co S is the convex hull of S, and $\sigma(T)$ is the spectrum of T. The following theorem is suggested by Prof. Z. Takeda:

Theorem 1. *An operator T is a convexoid if and only if the closed numerical range $\bar{W}(T)$ is spanned by the normal approximate*

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spectrum of T in the sense that

(3) $$\bar{W}(T)=\mathrm{co}\,\pi_n(T).$$

Proof. Suppose that T is a convexoid. Then an extreme point of $\bar{W}(T)$ belongs to $\sigma(T)$ and so $\pi(T)$, so that

$$\bar{W}(T)=\mathrm{co\ ext}\ \bar{W}(T)=\mathrm{co}\,(\sigma(T)\cap\partial\bar{W}(T)),$$

where ext S is the set of all extreme points of S and ∂S is the boundary of S. On the other hand, Hildebrandt's theorem [7; Satz 2] (a proof is contained in [5; Theorem 1]) implies that

$$\sigma(T)\cap\partial\bar{W}(T)\subset\pi_n(T).$$

Hence we have

$$\bar{W}(T)\subset\mathrm{co}\,(\sigma(T)\cap\partial\bar{W}(T))\subset\mathrm{co}\,\pi_n(T)\subset\bar{W}(T),$$

so that T satisfies (3).

Conversely, if T satisfies (3), then we have

$$\bar{W}(T)=\mathrm{co}\,\pi_n(T)\subset\mathrm{co}\,\sigma(T)\subset\bar{W}(T),$$

so that T satisfies (1).

In our previous note [3; Theorem 1], we have shown that $\lambda\in\pi_n(T)$ if and only if there is a character ϕ on the unital C^*-algebra $\mathfrak{A}$ generated by T satisfying

(4) $$\phi(T)=\lambda.$$

Hence Theorem 1 implies the following theorem which is originally suggested by Prof. Z. Takeda:

Theorem 2. *Let X be the character space of all characters of the unital C^*-algebra $\mathfrak{A}$ generated by T. Then T is a convexoid if and only if there exist "sufficiently many" characters of $\mathfrak{A}$ in the sense that*

(5) $$\bar{W}(T)=\mathrm{co}\,X(T),$$

where

(6) $$X(T)=\{\phi(T)\,;\,\phi\in X\}.$$

However, Theorem 2 does not mean that the state space $\sum$ of all states of $\mathfrak{A}$ is the convex closure (in the weak* topology) of X:

(7) $$\textstyle\sum=\overline{\mathrm{co}}\,X.$$

More precisely, we can show

Theorem 3. *An operator T has "sufficiently many" characters in the sense that the unital C^*-algebra $\mathfrak{A}$ generated by T satisfies* (7) *if and only if T is normal.*

Proof. If T is normal, then (7) is clearly satisfied which is implied by the well-known fact that the extreme points of the state space of an abelian C^*-algebra are multiplicative.

Suppose now that T is non-normal. Then $A=T^*T-TT^*\neq 0$ generates a non-trivial two-sided ideal $\mathfrak{R}$ which is called in [5; §5] the *pseudoradical* of $\mathfrak{A}$. By [5; Theorem 7], $\mathfrak{R}$ is the intersection of the kernels of all characters of $\mathfrak{A}$, so that $X(A)=\{0\}$. On the other hand, by a theorem of Berberian and Orland [1],

$$\bar{W}(A)=\sum(A), \tag{8}$$

so that $\sum(A)\neq\{0\}$; hence $\mathfrak{A}$ does not satisfy (7). This proves the theorem.

Remark. Mr. H. Takai presents an another proof of the necessity part of Theorem 3 without using Berberian-Orland's theorem. If (7) is satisfied, and if ϕ is a convex combination of characters, then we have

$$\phi(T^*T)=\phi(TT^*), \tag{9}$$

so that (9) is satisfied by any state ϕ by (7). By the completeness of the state space, we have $T^*T=TT^*$ from (9), that is, T is normal.

3. Tensor product. The tensor product of operators is recently discussed by many authors. Brown and Pearcy [2] show that

$$\sigma(A)\sigma(B)=\sigma(A\otimes B) \tag{10}$$

for any A and B. However, the equality (10) does not hold for other spectra. We shall show here

Theorem 4. *We have*

$$\pi_n(A)\pi_n(B)\subset\pi_n(A\otimes B). \tag{11}$$

Proof. Our proof is essentially same with the proof of Berberian in [8] for the approximate spectrum. Since

$$A\otimes B-\alpha\beta=(A-\alpha)\otimes\beta+A\otimes(B-\beta), \tag{12}$$

we have

$$\begin{aligned}\|(A\otimes B-\alpha\beta)x_n\otimes y_n\| &\leqq \|(A-\alpha)x_n\otimes y_n\|+\|Ax_n\otimes(B-\beta)y_n\|\\ &=\|(A-\alpha)x_n\|\|y_n\|+\|Ax_n\|\|(B-\beta)y_n\|\\ &=\|(A-\alpha)x_n\|+\|A\|\|(B-\beta)y_n\|\to 0\end{aligned}$$

and

$$\|(A\otimes B-\alpha\beta)^*x_n\otimes y_n\|\leqq\|(A-\alpha)^*x_n\|+\|A^*\|\|(B-\beta)y_n\|\to 0$$

as $n\to\infty$, if $\alpha\in\pi_n(A)(\beta\in\pi_n(B))$ and $\{x_n\}(\{y_n\})$ is a sequence of unit vectors satisfying $(*)$ and $(**)$ for A and α (B and β, respectively). Hence $\alpha\beta\in\pi_n(A\otimes B)$.

Remark. We can prove the theorem basing on the reciprocity of the characters of $\mathfrak{A}$ and $\pi_n(T)$ proved in [3; Theorem 1]. Let $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ be the unital C^*-algebras generated by A, B and $A\otimes B$ respectively. If ϕ and φ are characters on $\mathfrak{A}$ and $\mathfrak{B}$ respectively such that $\phi(A)=\alpha$ and $\varphi(B)=\beta$. Then we can easily check that $\phi\otimes\varphi$ is a character on the tensor product $\mathfrak{A}\otimes\mathfrak{B}$ of $\mathfrak{A}$ and $\mathfrak{B}$. Since $\mathfrak{A}\otimes\mathfrak{B}$ contains $\mathfrak{C}$, the restriction ψ of $\phi\otimes\varphi$ on $\mathfrak{C}$ is a character of $\mathfrak{C}$ satisfying $\psi(A\otimes B)=\alpha\beta$, so that $\alpha\beta\in\pi_n(A\otimes B)$ by [3; Theorem 1].

Theorem 5. *There are operators A and B for which the equality in* (11) *does not hold.*

Proof. Let A be a projection and B an operator whose normal approximate spectrum is empty. Then the left-hand side of (11) is empty. If $x\in\ker A$, then we have

$$(A\otimes B)x\otimes y=Ax\otimes By=0\otimes By=0$$

and

$$(A\otimes B)^*x\otimes y=A^*x\otimes B^*y=0\otimes B^*y=0,$$

so that the right-hand side of (11) contains 0 and is not empty. This proves the theorem.

In [5; § 3], we have observed that

$$T=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$$

has empty normal approximate spectrum. But,

$$T\otimes T=\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}$$

has a normal approximate propervalue, that is, $0\in \pi_n(T\otimes T)$. Therefore, we can use T and $T\otimes T$ as an example to prove Theorem 5.

In [5; Definition 1], we have introduced the class $\boldsymbol{S}$ of operators by

$$\boldsymbol{S}=\{T\in \mathfrak{B}(\mathfrak{H})\,;\,\pi_n(T)=\emptyset\}.$$

By the above T, we can conclude

Theorem 6. *There is an operator T which is not in $\boldsymbol{S}$ but the tensor product $T\otimes T$ belongs to $\boldsymbol{S}$.*

In general, the determinations of the approximate spectrum or the numerical range of the tensor product of operators are not easy, cf. [10]. In the notation of the preceeding remark, if every character of $\mathfrak{C}$ is extensible to a character of $\mathfrak{A}\otimes\mathfrak{B}$, especially if $A\otimes 1$ and $1\otimes B$ are contained in $\mathfrak{C}$, then $\pi_n(A)\pi_n(B)=\pi_n(A\otimes B)$, by the reciprocity of the characters and the normal approximate spectrum.

4. Applications. We shall give here two applications of Theorem 1. We shall begin with the following theorem [6; Theorem 3]:

Theorem 7 (Furuta-Nakamoto). *Let T be an invertible convexoid with the polar decomposition $T=UR$ and U be "cramped" in the sense that $\sigma(U)$ is contained in an open half-plane F with $0\notin F$. Then $0\notin \bar{W}(T)$.*

Proof. As noted in [5; § 4 Remark], the hypothesis insures us that $U,R\in\mathfrak{A}$ where $\mathfrak{A}$ is the unital C^*-algebra generated by T, and

$$\phi(T)=\phi(UR)=\phi(U)\phi(R)$$

for any character ϕ of $\mathfrak{A}$. Since $\phi(U)\in\sigma(U)\subset F$ by [3; Theorem 1] and $\phi(R)>0$, we have $\phi(T)\in F$. Hence $\pi_n(T)\subset F$ by a theorem of Kasahara-Takai [9], so that $\bar{W}(T)\subset F$ by Theorem 1, or $0\notin \bar{W}(T)$.

Remark. The converse of Theorem 7 is proved in [6; Theorem 6]. A sharpening of Theorem 7 is contained in [6*].

Mr. M. Enomoto kindly points out an alternative proof of Theorem 7: If $0\in\bar{W}(T)$, then by Theorem 1 there are characters $\phi_1,\cdots,\phi_n$ of $\mathfrak{A}$ and $\alpha_1,\cdots,\alpha_n$ with $\alpha_i\geqq 0$ and $\alpha_1+\cdots+\alpha_n=1$, and

$$0=\sum_{i=1}^{n}\alpha_i\phi_i(T)=\sum_{i=1}^{n}\alpha_i\phi_i(R)\phi_i(U)\in F,$$

so that we have a contradiction.

An alternative proof of [4; Theorem 2] is possible by Theorem 1:

Theorem 8. *For any operator A, there is a convexoid B such that $A\oplus B$ is a convexoid.*

Proof. Clearly, there is a convexoid B with $\bar{W}(A)\subset\bar{W}(B)$. By Theorem 1, we have

$$\begin{aligned}\bar{W}(A\oplus B)&=\mathrm{co}\,(\bar{W}(A)\cup\bar{W}(B))=\bar{W}(B)=\mathrm{co}\,\pi_n(B)\\&\subset\mathrm{co}\,\pi_n(A\oplus B)\subset\bar{W}(A\oplus B),\end{aligned}$$

so that $A\oplus B$ is a convexoid by Theorem 1.

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