85. On Geometrical Classification of Fibers in Pencils of Curves of Genus Two

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All possible numerical types of fibres in pencils of curves of genus two have been classified by Ogg ([9])**) and Iitaka ([5]) independently, and recently Winters has shown that all of them arise actually as a corollary of his more general existence theorem ([12]).

This article contains their geometrical classification which is essentially different from the numerical one. By using our method it can be also shown that all possible types arise. Our method should be generalized for pencils of curves with arbitrary genus. The complete classification and explicit construction of fibres will be shown in our forthcoming papers ([7], [8]).

1. Construction of geometrical invariants and characterization of fibres by them.

(1) Let $\pi: X \to D$ be a pencil of (complete) curves of genus two over a disc $D = \{t \in \boldsymbol{C}; |t| < \varepsilon\}$. Further, we assume the following:

i) X is a non-singular (complex analytic) surface free from exceptional curves of the first kind;

ii) π is smooth over a punctured disc $D' = D - \{0\}$. Thus for every $t \in D'$ the fibre $X_t = \pi^{-1}(t)$ is a complete non-singular curve of genus two.

In this article we consider only such pencils.

(2) (See [10] for detailed discussion in this paragraph.) For every $t \in D'$ denote by J_t the jacobian variety of X_t. Then the J_t's form a holomorphic family $\boldsymbol{J}$ of abelian varieties of dimension two over D'. Moreover $\boldsymbol{J}$ is a polarized bundle in the sense of [10]. Therefore we can construct a canonical multivalued holomorphic map $T_\pi: D' \to \mathfrak{S}_2$ where $\mathfrak{S}_2$ is the Siegel upper-half plane of degree two.

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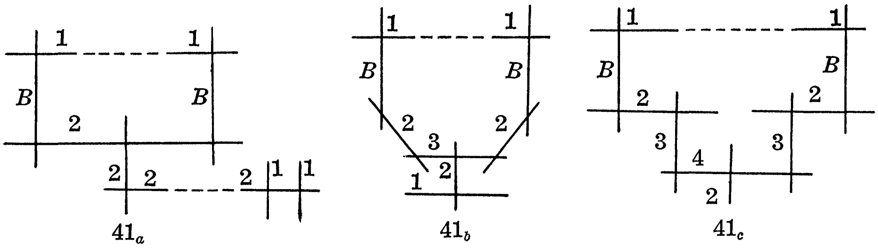
**) In Ogg's table the following types are missing.



41_a 41_b 41_c

Let $T_\pi(\gamma t)$ be the analytic continuation of $T_\pi(t)$ along a circle in D' rounding the origin once counterclockwise and whose base point is t. Then, for some element M_π of the symplectic group $Sp(2, \boldsymbol{Z})$, we have

$$(*) \qquad T_\pi(\gamma t) = M_\pi T_\pi(t)$$

where M_π operates on $\mathfrak{S}_2$ in a well-known way.

Two (multivalued) holomorphic maps $T_i : D' \to \mathfrak{S}_2 (i=1, 2)$ are called equivalent if there is an element M of $Sp(2, \boldsymbol{Z})$ such that $T_1(t) = MT_2(t)$ for every $t \in D'$. In the above construction T_π can not be defined uniquely but the equivalence class of T_π depends only on π. The conjugate class of M_π also depends only on π.

Definition (3). i) The holomorphic map T_π above is called the characteristic map of π.

ii) The conjugate class of M_π (which we denote by the same letter M_π) is called the Picard-Lefschetz transformation (or monodromy) of π.

(4) In a suitable compactification $\bar{\mathfrak{S}}_2^*$ of $\mathfrak{S}_2^* = \mathfrak{S}_2 / Sp(2, \boldsymbol{Z})$ we can consider the limit point

$$Z_\pi = \lim_{t \to 0} (T_\pi(t) \text{ mod. } Sp(2, \boldsymbol{Z})).$$

Note that $Z_\pi \in \mathfrak{S}_2^*$ if and only if M_π is of finite order and in this case Z_π is a fixed point of M_π.*) Here we use the compactification due to Igusa ([4]).**) The boundary $\mathscr{B}_0$ of $\bar{\mathfrak{S}}_2^*$ is isomorphic to $\boldsymbol{P}_1 \times \boldsymbol{P}_1$ which is a trivial bundle over $\boldsymbol{P}_1$ by the first projection $p ; \boldsymbol{P}_1 \times \boldsymbol{P}_1 \to \boldsymbol{P}_1$ This base space $\boldsymbol{P}_1$ should be considered as the compactification of the moduli space of elliptic curves $\boldsymbol{P}_1 = \mathfrak{S}_1 / SL(2, \boldsymbol{Z}) \cup \{\infty\}$. Let $\mathscr{B} = \mathscr{B}_0 - p^{-1}(\infty)$, $\mathcal{C} = p^{-1}(\infty) - \{(\infty, \infty)\}$ and $\mathscr{D} = (\infty, \infty)$.

On $\mathfrak{S}_2$ considered as the moduli space of principally polarized abelian varieties of dimension two, those points which correspond to products of two elliptic curves with the canonical polarization form a closed submanifold $\mathfrak{N}$ of codimension one. Let $\mathscr{M} = \mathfrak{S}_2 - \mathfrak{N}$.

It is known that M^n is unipotent for some n. Take the least such n. Let $E' = \{s ; 0 < |s| < \varepsilon^{1/n}\}$ and let μ_n be a map defined by

$$\begin{array}{ccc} \mu_n : E' & \longrightarrow & D' \\ \Psi & & \Psi \\ s & \longrightarrow & s^n. \end{array}$$

Then as the image of D' by T_π is cotained in $\mathscr{M}$, the (local) mapping degree deg π of $T_\pi \circ \mu_n : E' \to \mathscr{M}$ at the origin can be defined.

Definition (5). Let $\pi : X \to D$ be a pencil of curves of genus two and let X_0 be the fibre at the origin.

i) If M_π is of finite order, we call X_0 of elliptic type. If M_π is of infinite order, we call X_0 of parabolic type.

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*) Such Z_π's have been classified by Gottschling ([2], [3]) and the corresponding hyperelliptic curves by Bolza ([1]). The conjugate classes of M_π's of finite order have been classified by the second author ([10], revised in [11]).

**) In this case of $\mathfrak{S}_2^*$ this compactification is originally due to Satake [13].

ii) Let X_0 be of elliptic type. We call X_0 of type [1] if $Z_\pi \in \mathcal{M}$ (in this case $\deg \pi = 0$) and of type [2] if $Z_\pi \in \mathcal{N}$.

iii) Let X_0 be of parabolic type. We call X_0 of type [3] if $Z_\pi \in \mathcal{B}$, of type [4] if $Z_\pi \in \mathcal{C}$ and of type [5] if $Z_\pi = \mathcal{D}$.

We have the following theorems.

Theorem (6). *If there is given a multivalued holomorphic map $T: D' \to \mathcal{M}$ satisfying $(*)$, we can construct a pencil of curves of genus two $\pi: X \to D$ whose characteristic map T_π is equivalent to T.*

Theorem (7). *If two pencils $\pi_i: X_i \to D (i=1, 2)$ have characteristic maps equivalent to each other, then for some smaller open disc E containing the origin there is an isomorphism i between the restrictions of X_i's over E which is compatible with π_i's.*

$$\begin{array}{ccc} X_1|E & \xrightarrow{i} & X_2|E \\ {\scriptstyle \pi_1}\downarrow & & {\scriptstyle \pi_2}\downarrow \\ E & = & E \end{array}$$

Theorem (8). *Let $\pi: X \to D$ be a pencil of curves of genus two. Then the fibre at the origin $X_0 = \{\pi = 0\}$ (considered as a divisor) depends only on Z_π, M_π and $\deg \pi$.*

Corollary (9). *The topological model of X_0 depends only on M_π and $\deg \pi$. Its homotopy type depends on M_π only.*

2. New phenomena. In this section we show some new phenomena which do not occur in the case of pencils of elliptic curves which have been studied deeply by Kodaira ([6]).

(1) There are singular fibres whose Picard-Lefschetz transformations are trivial. More precisely two elliptic curves joined by a series of projective lines (Fig. 1) have trivial monodromy. This is of type [2] and $\deg \pi - 1$ is equal to the number of projective lines.

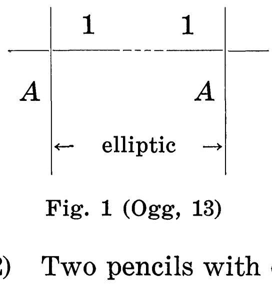

Fig. 1 (Ogg, 13)

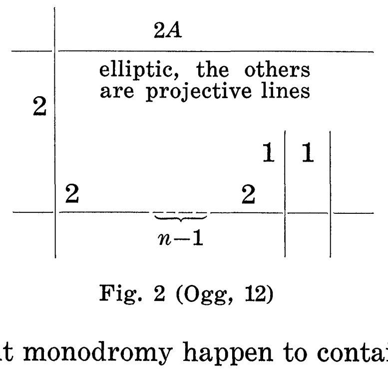

Fig. 2 (Ogg, 12)

(2) Two pencils with different monodromy happen to contain the same singular fibre. For example a singular fibre, as in Fig. 2, arises in two pencils with the following invariants:

i) $Z_\pi=\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$, $M_\pi=\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

$\deg \pi=2n$ (This is of type [2].);

ii) $Z_\pi=\begin{pmatrix} z & \frac{1}{2}z \\ \frac{1}{2}z & \infty \end{pmatrix}$, $M_\pi=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & -n \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

$\deg \pi=0$. (This is of type [3].)

(3) Some types have no components with simple multiplicity though they are not multiple fibres. They are of type 17, 18, and 30 in the sense of Ogg's classification ([9]).

When we construct pencils with given invariants as pencils of binary quintics or sextics $y^2=f(x,t)$ with parameter t, these types have to be written in the form

$$y^2=tg(x,t).$$

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