

82. On Representations of Homology Classes

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1. Introduction. R. Thom [3] has shown that every integral $(n-1)$ -dimensional homology class θ of an orientable n -manifold M is representable by an $(n-1)$ -submanifold of M . In this result the submanifold representing θ is not required to be connected. In the present paper, we shall consider under what condition θ is representable by a connected $(n-1)$ -submanifold. Our result is stated as follows:

Theorem. *Let M be a compact connected orientable manifold of dimension $n \geq 3$ with connected boundary (possibly empty). Let $\{g_1, \dots, g_r\}$ be a free basis for the group $H_{n-1}(M; \mathbb{Z})$. Then, for a non-zero homology class*

$$\theta = a_1 g_1 + \dots + a_r g_r (a_i \in \mathbb{Z}),$$

the following conditions are mutually equivalent;

- (i) θ can be represented by a connected $(n-1)$ -submanifold.
- (ii) The greatest common divisor $(|a_1|, \dots, |a_r|)$ is 1.
- (iii) There is a homology class $\alpha \in H_1(M; \mathbb{Z})$ such that the intersection $\theta \cdot \alpha$ is 1.

Everything will be considered from the *PL* viewpoint. However we note that the similar argument is applicable in the differentiable viewpoint. I am grateful to Mr. K. Yokoyama for his suggestions given me at the very beginning of this work.

2. Attaching handles. Throughout this paper all manifolds, with or without boundary, are to be compact, oriented and *PL*. All submanifolds of a manifold M are, moreover, to be closed and locally flat in M . The boundary of a manifold M is denoted by ∂M and the interior of M by $\text{int } M$. The manifold M with orientation reversed is denoted by $-M$.

Let A be an $(n-1)$ -submanifold of an n -manifold M and let $f: I = [0, 1] \rightarrow M$ be a simple arc. When $f(I)$ meets A transversely at $P \in M$, the intersection number of A and f at P , denoted by $\text{sign}(A, f; P)$, is defined as follows: Since f is transversal to A at P , there exists a *PL* homeomorphism $h: U \rightarrow B^{n-1} \times B^1$ so that $h(P) = (0, 0)$, $h(U \cap A) = B^{n-1} \times 0$ and $h(U \cap f(I)) \subset 0 \times B^1$, where U is a ball neighborhood of P in M and B^i denotes a *PL* i -ball and 0 denotes an interior point corresponding to the barycenter of the standard simplex. Choose h so that $U \cap A$, $U \cap f(I)$ are mapped with natural orientation. We then define

$\text{sign}(A, f: P) = +1$ or -1 according as whether U is mapped with correct orientation or not.

Now let $\omega: \partial B^2 \rightarrow M$ be a simple loop which is transversal to an $(n-1)$ -submanifold A . For each point P in $A \cap \omega(\partial B^2)$, we choose an orientation preserving PL imbedding $\varphi_p: I \rightarrow \partial B^2$ such that $\omega^{-1}(P) \in \text{int } \varphi_p(I)$, and write $\text{sign}(A, \omega: P)$ for $\text{sign}(A, \omega \cdot \varphi_p: P)$. Then the intersection number of A and ω , denoted by $\text{sign}(A, \omega)$, is defined by

$$\text{sign}(A, \omega) = \sum_P \text{sign}(A, \omega: P)$$

where P runs over all points in $A \cap \omega(\partial B^2)$.

Let A be an $(n-1)$ -submanifold of an n -manifold M and let $F: B^{n-1} \times I \rightarrow M$ be an orientation preserving PL imbedding such that $F(B^{n-1} \times I) \cap A = F(B^{n-1} \times \partial I)$. Assume that the sum of the intersection numbers

$$\text{sign}(A, f: f(0)) + \text{sign}(A, f: f(1))$$

is zero for a simple arc $f: I \rightarrow M$ defined by $f(t) = F(0, t)$ ($t \in I$). Then, as proved in the following, we obtain an $(n-1)$ -submanifold

$$A_* = \{A - \text{int } F(B^{n-1} \times \partial I)\} \cup F(\partial B^{n-1} \times I)$$

with orientation compatible with that of A , and A_* is L -equivalent to A in the sense of Thom [3]. We shall say that A_* is obtained from A by attaching a 1-handle along a simple arc f .

First take a bicollar neighborhood $B = A \times [-1, 1]$ in M ([2]). Let K be a triangulation of M such that $F(B^{n-1} \times I)$, $A \times [-1, 0]$, $A \times [0, 1]$ are subcomplexes L, K_-, K_+ of K . We may assume that these subcomplexes are full in K . Let $K_0 = K_- \cap K_+$, then we have $|K_0| = A$ and $|\overline{K_0 - K_0} \cap \overline{L} \cup L_1| = A_*$ with $|L_1| = F(\partial B^{n-1} \times I)$. Note that $K, L, K_-, K_+, L \cap K_0$ and L_1 are all combinatorial manifolds. Let v be a vertex of L_1 in $F(\partial B^{n-1} \times \partial I)$. We may assume that $K_- \cap L = K_0 \cap L$ without loss of generality. Then $|Lk(v: K_- \cup L)|$ is a PL $(n-1)$ -ball, because $|Lk(v: K_- \cup L)|$ is a union of PL $(n-1)$ -balls $|Lk(v; K_-)|$, $|Lk(v: L)|$ which intersect at PL $(n-2)$ -ball $|Lk(v: K_0 \cap L)|$ contained in the common combinatorial face of $|Lk(v: K_-)|$ and $|Lk(v: L)|$ ([1]). Therefore a pair $(|Lk(v: K)|, |Lk(v: \overline{K_0 - K_0} \cap \overline{L} \cup L_1)|)$ is an unknotted sphere pair. Thus A_* is a compact closed locally flat $(n-1)$ -submanifold of M . Put

$$W^n = \{A \times [0, 1] - F(B^{n-1} \times \partial I)\} \cup F(\partial B^{n-1} \times I) \times \left[0, \frac{1}{2}\right] \cup F(B^{n-1} \times I) \times \left\{\frac{1}{2}\right\}.$$

Then W^n gives an unoriented locally flat L -equivalence in $M \times [0, 1]$ between A_* and A . The condition

$$\text{sign}(A, f: f(0)) + \text{sign}(A, f: f(1)) = 0$$

permits us to orient W^n .

Lemma 1. *Let A_1, A_2 be disjoint connected $(n-1)$ -submanifolds of a connected manifold M of dimension $n \geq 3$, and assume that there*

is a simple loop ω in M which meets A_1 transversely at a single point $\omega(0)=\omega(1)^*$ and does not meet A_2 , then the disjoint sum A_1+A_2 is L -equivalent to a connected submanifold.

Proof. Since $M-A_1$ is connected, there exists a simple arc $a: I \rightarrow M$ satisfying

$$\begin{aligned} a(I) \cap A_2 &= \{a(0)\}, \\ a(I) \cap \omega(\partial B^2) &= \left\{ a(1) = \omega\left(\frac{1}{2}\right) \right\}, \\ a(I) \cap A_1 &= \emptyset. \end{aligned}$$

Consider simple arcs f_1, f_2 defined by

$$\begin{aligned} f_1(t) &= \begin{cases} a(2t) & \text{for } 0 \leq t \leq \frac{1}{2}, \\ \omega(t) & \text{for } \frac{1}{2} \leq t \leq 1; \end{cases} \\ f_2(t) &= \begin{cases} a(2t) & \text{for } 0 \leq t \leq \frac{1}{2}, \\ \omega(1-t) & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases} \end{aligned}$$

We have obviously

$$\text{sign}(A_1, f_1 : f_1(1)) + \text{sign}(A_1, f_2 : f_2(1)) = 0$$

and hence we may assume that

$$\text{sign}(A_1, f_1 : f_1(1)) + \text{sign}(A_2, f_1 : f_1(0)) = 0.$$

Choose a suitable PL imbedding $F: B^{n-1} \times I \rightarrow M$ satisfying $F(B^{n-1} \times I) \cap (A_1 + A_2) = F(B^{n-1} \times \partial I)$, $F(0, t) = f_1(t)$ ($t \in I$). Then we get the connected submanifold $(A_1 + A_2)_*$ which is L -equivalent to $A_1 + A_2$.

Lemma 2. *Let A be an $(n-1)$ -submanifold in a connected manifold M of dimension $n \geq 3$, and assume that there is a loop ω in M which meets A transversely at a single point. Then A is L -equivalent to a connected submanifold.*

Proof. In virtue of Lemma 1, induction on the number of components of A proves Lemma 2.

3. Representation. Let M be an n -manifold, and $\theta \in H_{n-1}(M; Z)$ be an integral homology class. We say that θ is representable if there exists an $(n-1)$ -submanifold A such that $\theta = i_*([A])$, where i_* is the homology map induced by the inclusion map $i: A \rightarrow M$ and $[A]$ is the fundamental class of A . The PL analogue of the arguments in Thom [3] shows that every θ is representable by an $(n-1)$ -submanifold of M and that L -equivalent submanifolds of M represent the same homology class.

Lemma 3. *Let M be a connected n -manifold, $n \geq 3$. Let $\theta \in H_{n-1}(M; Z)$ be a class such that the intersection number $\theta \cdot \alpha$ is 1 for*

*) Here we identify $B^2 = I/\partial I$.

some $\alpha \in H_1(M; Z)$. Then there exists a submanifold A representing θ and a simple loop ω representing α such that ω meets A at a single point.

Proof. Take an $(n-1)$ -submanifold A' representing θ . Furthermore take a simple loop ω representing α . We may assume that ω meets A' transversely, and hence ω meets A' at finitely many points $P_1 = \omega(t_1), \dots, P_r = \omega(t_r) (t_1 < \dots < t_r)$. We shall construct by induction on r a submanifold A which meets $\omega(\partial B^2)$ at a single point. We may assume $r \geq 3$. Using the well-known fact that the homological intersection number coincides with the geometrical one, we have

$$\text{sign}(A', \omega) = \sum_{i=1}^r \text{sign}(A', \omega : P_i) = 1.$$

Therefore, for some i we have

$$\text{sign}(A', \omega : P_i) + \text{sign}(A', \omega : P_{i+1}) = 0.$$

Now, by attaching a 1-handle to A' along the simple subarc $\omega|_{[t_i, t_{i+1}]}$, we obtain A'_* which meets ω at $(r-2)$ points. Then A'_* has the inductive property.

Lemma 4. Let M be an n -manifold with connected boundary (possibly empty), $n \geq 3$ and let $\theta \in H_{n-1}(M; Z)$ be a non-zero homology class which is represented by a connected $(n-1)$ -submanifold A . Then there exists a homology class $\alpha \in H_1(M; Z)$ such that $\theta \cdot \alpha$ is 1.

Proof. Let M_0 be the component of M containing A . The submanifold A has a bicollar neighborhood $B = A \times [-1, 1]$ in M_0 . By removing $\text{int } B$, we get an n -manifold $W = M_0 - \text{int } B$ with $\partial W = A + (-A) + \partial M_0$. If W is disconnected, then A is obviously L -equivalent to zero in M , and hence we have $\theta = 0$ which is a contradiction. Thus W is connected. Therefore there exists a simple arc f in $W - M_0$ such that $f(0) \in A \times \{1\}$, $f(1) \in A \times \{-1\}$ and $p(f(0)) = p(f(1))$ where $p: A \times [-1, 1] \rightarrow A$ is the projection. Consider a simple loop $\tilde{f}: \partial B^2 \rightarrow M$ defined by

$$\tilde{f}(t) = \begin{cases} f(2t) & \text{for } 0 \leq t \leq \frac{1}{2}, \\ (p(f(0)), 4t-3) & \text{for } \frac{1}{2} \leq t \leq 1, \end{cases}$$

and let α be the homology class represented by $\tilde{f}(\partial B^2)$. Then it holds that $\theta \cdot \alpha = 1$.

We can now prove the theorem. Lemma 2 and Lemma 3 show (iii) \Rightarrow (i), and Lemma 4 shows (i) \Rightarrow (iii). It is easy to check that (ii) is equivalent to (iii).

References

- [1] J. F. P. Hudson: Piecewise Linear Topology. Benjamin, New York (1969).
- [2] Morton Brown: Locally flat imbeddings of topological manifolds. Ann. of Math., **75**(2), 331-341 (1962).
- [3] R. Thom: Quelques propriétés globales des variétés différentiables. Comm. Math. Helv., **28**, 17-86 (1954).