81. *Qualitative Theory of Codimension-one Foliations*

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We shall give a method of studying topological properties of integral manifolds of a completely integrable one-form.

Suppose that we are given a connected, closed $(n+1)$-manifold V^{n+1} of class C^4 with a nonsingular, completely integrable one-form ω of class $C^3, n\geqslant 1$. As in [1], a maximal connected integral manifold of ω will be called a leaf.

1. The critical cycles $\boldsymbol{\Sigma}$. For each $p\in V$, by assumption, there is a local coordinate system $(x^1,\cdots,x^{n+1})$ of class C^3 in a neighborhood U of p such that $\omega|U=fdx^{n+1}$ for some positive-valued C^3 function f on U. Then the set $(U,f,(x^1,\cdots,x^{n+1}))$ is called an *$\mathcal{F}$-chart* (at p). Denote by Σ the set of zeros of the exterior derivative of ω, i.e., $\Sigma=\{p\in V\,|\,(d\omega)_p=0\}$.

Let $p\in\Sigma$. Let $(U,f,(x^1,\cdots,x^{n+1}))$ be an $\mathcal{F}$-chart at p and put

$$j^2_x(f)=\left(f_{ij}(x)\,;\,\begin{matrix} i\downarrow 1,\cdots,n\\ j\rightarrow 1,\cdots,n\end{matrix}\right),$$

$$j^3_x(f)=\left(f_{ij}(x),\frac{\partial}{\partial x^i}\det j^2_x(f)\,;\,\begin{matrix} i\downarrow 1,\cdots,n+1\\ j\rightarrow 1,\cdots,n\end{matrix}\right),$$

where $f_{ij}(x)=\partial^2 f(x)/\partial x^i\partial x^j$. Let $i=0,1,\cdots,n$. The point p is said to be of *type* (i) if the matrix $j^2_p(f)$ is nonsingular and if the number of negative eigenvalues of $j^2_p(x)$ is equal to i. We say that p is of *type* $(*)$ if $\det j^2_p(f)=0$. Of course, the type of a point of Σ is well defined independently of the choice of $\mathcal{F}$-charts. For $\lambda=0,1,\cdots,n$ or $*$, let Σ_λ denote the set of points of type (λ). Then we have $\Sigma=\Sigma_*\cup\Sigma_0\cup\cdots\cup\Sigma_n$ (disjoint union).

We shall assume that ω satisfies the following condition:

(T) *For any $p\in\Sigma_*$, there is an $\mathcal{F}$-chart $(U,f,(x^1,\cdots,x^{n+1}))$ at p such that the matrix $j^3_p(f)$ is nonsingular.*

One sees then that the same condition holds for any $\mathcal{F}$-chart at $p\in\Sigma_*$. One will also see that this condition (T) is "generic".

2. The main theorems. Assume that ω satisfies the condition (T). Then we have the following three theorems.

Theorem I. *If $\Sigma_0\neq\emptyset$ and $\Sigma_1=\emptyset$, then there exists a C^3 fibre bundle B^{n+1} over S^1 and a C^3 diffeomorphism $h: B^{n+1}\rightarrow V^{n+1}$ such that*

(i) *the fibre of B^{n+1} is a connected, simply connected, closed n-manifold of class C^3.*

(ii) *for each fibre M^n of B^{n+1}, h induces a C^3 diffeomorphism of M^n onto a leaf of V.*

Theorem II. *If $\Sigma_n \neq \emptyset$, $\Sigma_{n-1} = \emptyset$ and $\Sigma_0 = \emptyset$, then for any $p \in \Sigma_n$, there exists a C^3 fibre bundle R^{n+1} over S^1 and a C^3 imbedding $h: R^{n+1} \to V^{n+1}$ such that*

(i) *the fibre of R^{n+1} is a connected, simply connected, noncompact n-manifold of class C^3, without boundary.*

(ii) *for each fibre N^n of R^{n+1}, h induces a C^3 diffeomorphism of N^n onto a leaf of V.*

(iii) *there exist a finite number of compact leaves $K_1, K_2, \cdots, K_k$, $1 \leqslant k < \infty$, such that $K_1 \cup \cdots \cup K_k = \overline{h(R^{n+1})} - h(R^{n+1}) = \overline{L(p)} - L(p)$, where $L(p)$ is the leaf through p.*

(iv) *$h(R^{n+1}) \cap \Sigma_n =$ the connected component of Σ containing p.*

Theorem III. *If $\Sigma_n = \emptyset$, then there exists an open dense subset V_0 of V such that for any $p \in V_0$, the leaf through p is locally everywhere dense in the sense of Reeb* [1, p. 108].

Detailed proofs will appear elsewhere.

3. The veins. Let X be a C^3 vector field satisfying $\omega(X) = 1$. Put $\omega' = -\mathcal{L}_X \omega$, where $\mathcal{L}_X$ denotes the Lie derivative with respect to X.

Lemma 3.1. *For an $\mathcal{F}$-chart $(U, f, (x^1, \cdots, x^{n+1}))$, we have*

$$\omega' \mid U = \textstyle\sum_{i=1}^n (\partial \log f / \partial x^i) dx^i + (-X(f) + \partial \log f / \partial x^{n+1} dx^{n+1}).$$

This implies that for a leaf L, $\omega'|_L$ is a closed one-form on L and is defined independently of the choice of X.

Definition 3.1. Let J^{n-1} be a connected, closed $(n-1)$-submanifold of class C^3 in V. J^{n-1} is called a compact *vein* (without singularity) of (V, ω) if $\omega_x(v) = 0$ and $\omega'_x(v) = 0$ for all $x \in J^{n-1}$ and $v \in T_x(J^{n-1})$.

4. Closed one-forms and Morse theory. Let M^n be a connected complete Riemannian n-manifold of class C^3, without boundary, and let α be a closed one-form of class C^2 such that every singular point is nondegenerate. For a singular point p of α, the *index* of p is defined to be the number of negative eigenvalues of the Jacobian matrix of α at p. α is said to be *proper* if the dual vector field α^* of α is complete and if there exist two families $\{E_i\}_{i \in I}, \{\tilde{E}_i\}_{i \in I}$ of open sets of M such that

(i) for a singular point p of α, there is $i \in I$ such that $p \in E_i \subset \tilde{E}_i$.

(ii) there exist three positive constants a_0, b_0, and c_0 such that (a) $\|\alpha_x\| > a_0$ for all $x \in M - \bigcup_{i \in I} E_i$, (b) $\mathrm{dis}(E_i, M - \tilde{E}_i) > b_0$ for all $i \in I$, and (c) $\mathrm{diam}(\tilde{E}_i) < c_0$ for all $i \in I$.

Proposition 4.1. *Suppose that α is proper and has at least one singular point of index 0. Let p be the singular point of index 0. If α has no singular point of index 1, then*

(i) *there exists a C^3 function $f: M^n \to \boldsymbol{R}$ which is proper, such that $\alpha = df$.*

(ii) *M^n is simply connected.*

(iii) *if α has no singular point of index n, then M^n is noncompact.*

(iv) *the number of the singular points of index 0 is equal to one.*

(v) *for any $x \in M$, there exists a piecewise C^1 curve $\mathfrak{a}: [0,\tau] \to M^n$ which satisfies $\mathfrak{a}(0)=p$, $\mathfrak{a}(\tau)=x$, such that $\dot{\mathfrak{a}}(t)=\alpha^*_{\mathfrak{a}(t)}/\|\alpha^*_{\mathfrak{a}(t)}\|$ for $t \in [0,\tau]$ where $\alpha^*_{\mathfrak{a}(t)} \neq 0$.*

This is a generalization of Reeb [1, (C, I, 9)]. Another generalization will be given in [2].

5. Lifts of tangential curves. From now on, fix a C^3 Riemann metric g of V and a C^3 vector field X satisfying $\omega(X)=1$. Let $\omega' = -\mathcal{L}_X \omega$. Denote by $\{\varphi_s\}_{s \in \boldsymbol{R}}$ the one-parameter group of transformations generated by X. A continuous curve $\mathfrak{c}: [0,\tau] \to V$ is called *tangential* if the image of $\mathfrak{c}$ is contained in a leaf. For a tangential curve $\mathfrak{c}$ and $\eta \in \boldsymbol{R}$, suppose that there is a continuous function $\sigma: [0,\tau] \to \boldsymbol{R}$, $\eta=\sigma(0)$, such that the curve $\mathfrak{d}: [0,\tau] \to V$ defined by $\mathfrak{d}(t)=\varphi_{\sigma(t)}(\mathfrak{c}(t))$, is tangential. Then $\mathfrak{d}$ is called the *η-lift* of $\mathfrak{c}$ and σ is called the *height parameter* of the η-lift of $\mathfrak{c}$.

Lemma 5.1. *Suppose that a C^1 tangential curve $\mathfrak{c}$ has the η-lift for some η. Then the height parameter $s=\sigma(t)$ satisfies the following differential equation: $ds/dt = -\omega(\varphi_{s*}(\dot{\mathfrak{c}}(t)))$ with initial condition $s(0)=\eta$.*

Now, choose a constant κ so that

$$\kappa > \left| \frac{\partial^2}{\partial s^2} \omega(\varphi_{s*} v) \right| \qquad \text{for all } v \in T_1(V)$$

and all s satisfying $|s| \leq 1$, where $T_1(V)$ denotes the tangent sphere bundle of V^{n+1}.

Lemma 5.2. *Let $\mathfrak{c}: [0,\tau] \to V$ be a C^1 tangential curve such that $\omega'(\dot{\mathfrak{c}}(t))=0$ and $\|\dot{\mathfrak{c}}(t)\|=1$ for all $t \in [0,\tau]$. If $|\eta| < 1/(\kappa\tau+1)$, then $\mathfrak{c}$ has the η-lift.*

6. Admissible tangential curves. Let Y be the vector field on V defined by the formulas $\omega(Y)=0$, $\omega'(v)=g(Y,v)$ for all $v \in \omega^{-1}(0)$.

Definition 6.1. A tangential curve $\mathfrak{a}: [0,\tau] \to V$ is *admissible* if $\mathfrak{a}$ is piecewise C^1 and if $\dot{\mathfrak{a}}(t) = -Y_{\mathfrak{a}(t)}/\|Y_{\mathfrak{a}(t)}\|$ for $t \in [0,\tau]$ satisfying $Y_{\mathfrak{a}(t)} \neq 0$.

Lemm 6.1. *Suppose that ω satisfies the condition* (T). *Then there exist positive constants α_*, τ_*, such that*

$$\int_{\mathfrak{a}[0,\tau]} \omega' < -\alpha_* \tau$$

for any admissible tangential curve $\mathfrak{a}: [0,\tau] \to V$ satisfying $\tau \geq \tau_*$.

Lemma 6.2. *Suppose that ω satisfies the condition* (T). *Then there exists positive constant A such that, if $0 < \eta < 1/(\kappa A+1)$, then any admissible tangential curve $\mathfrak{a}: [0,\tau] \to V$ has the η-lift and then the height parameter σ has the following estimate: $\sigma_-(t) < \sigma(t) < \sigma_+(t)$ for any $t \in [0,\tau]$, where $\sigma_\pm(t)$ is defined by*

$$\sigma_\pm(t) = \left(\exp \int_{\mathfrak{a}[0,t]} \omega' \right) \Big/ \left(\frac{1}{\eta} \mp \kappa A \right).$$

7. Notes on the proofs of theorems. In order to prove Theorem I, note the following lemma and use Reeb stability theorem [1, (B, III, 11)].

Lemma 7.1. *Under the same hypotheses of Theorem* I, *if a leaf* L *through a point* $p \in \Sigma_0$ *is noncompact, then for any* $\eta_* > 0$, $\tau_* > 0$, *there exist* $x \in L$, $\eta \in \boldsymbol{R}$ *satisfying* $0 < \eta < \eta_*$ *and an admissible tangential curve* $\alpha : [0, \tau] \to V$ *such that* $\varphi_\eta(x) \in L$, $\alpha(0) = x$, $\alpha(\tau) = p$ *and* $\tau \geq \tau_*$.

The essential part of the proof of Theorem II is the following lemma.

Lemma 7.2. *Suppose that there is a curve* $\alpha : [0, \infty) \to V$ *such that for each* $\tau \in [0, \infty)$, *the restriction* $\alpha|_{[0,\tau]}$ *is an admissible tangential curve. Then, under the same hypotheses of Theorem* II, *for almost every* $t \in [0, \infty)$ *there exists a compact vein* J_t *without singularity, such that* J_t *contains* $\alpha(t)$ *and that*

$$\mathrm{diam}_{J_t}(J_t) \cdot \sigma_-(t) \to 0 \qquad (t \to \infty).$$

For the proof of Theorem III, note the following lemma.

Lemma 7.3. *Under the same hypothesis of Theorem* III, *for any* $x \in V$ *and any* $\tau > 0$, *there exists an admissible tangential curve* $\alpha : [0, \tau] \to V$ *satisfying* $\alpha(\tau) = x$.

References

[1] G. Reeb: Sur certaines propriété topologiques des variétés feuilletées. Act. Sci. et Ind., Hermann, Paris (1952).

[2] K. Yamato: Codimension-one foliations with singularities (to appear).