110. *On the Propagation of Error in Numerical Integrations*

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(Comm. by Kenjiro SHODA, M. J. A., Sept. 12, 1972)

§0. Introduction. Even with quite simple differential equations, it can happen that their solutions are not expressible in a closed form and that a numerical approach is the most convenient way to deal with the problem. And in this case if an approximate value y_n of the solution $y(x)$ of a differential equation at the point x_n has been calculated by some approximate methods, the estimate on the magnitude of error

(0.1) $$e_n = y_n - y(x_n) \qquad (n=1,2,3,\cdots)$$

is of great importance.

While we possess simple and useful error estimate for the propagation of error, it seems, however, that if we concern with the problem of asymptotic behavior of the propagation of error, not so many results appeared. The purpose of this paper is to state some results on a propagation of error of some approximate equations.

Finally the author wishes to express his thanks to Dr. M. Yamamoto of Osaka University for his kind advices and invaluable suggestions.

§1. First we consider the first order differential equation:

(1.1) $$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0. \end{cases}$$

We shall now try to approximate the equation (1.1) by the difference equation:

(1.2) $$y_{n+1} = y_n + hf(x_n, y_n)$$

which is known as Euler's method.

In actual calculation, the calculated value of y_{n+1} is given by the formula:

(1.3) $$y_{n+1} = y_n + hf(x_n, y_n) - R_{n+1} \qquad (R_n\text{: round-off error})$$

On the other hand, if we denote the true value of the solution of (1.1) at the point $x = x_n$ by $y(x_n)$, we have also the relation:

(1.4) $$y(x_{n+1}) = y(x_n) + hf(x_n, y(x_n)) + T_{n+1},$$

where T_n denotes the truncation error corresponding to the n-th step. If we subtract (1.3) from (1.4) and write

(1.5) $$E_n = T_n + R_n,$$

we find the difference equation:

(1.6) $$e_{n+1} = e_n + h(f(x_n, y(x_n)) - f(x_n, y_n)) + E_{n+1}.$$

We notice first that we may write

$$f(x_n, y(x_n)) - f(x_n, y_n) = f_y(x_n, \eta_n)(y(x_n) - y_n)$$

if f_y exists, where η_n is a number between y_n and $y(x_n)$, so that (1.6) may be written in the form:

(1.7) $$e_{n+1} = e_n + he_n f_y(x_n, \eta_n) + E_{n+1}.$$

Here we discuss the asymptotic behavior of the solution of the difference equation (1.7).

At first we shall give several lemmas.

Lemma 1.1. *The solution of the difference equation:*

$$\nabla z(x_0+nh) = Az(x_0+(n-1)h) + B(x_0+(n-1)h)z(x_0+(n-1)h) + w(x_0+(n-1)h) \qquad (A:\ \text{constant})$$

is

$$z(x_0+nh) = \frac{z(x_0)}{1+A} Y(x_0+nh) + Y(x_0+nh) \sum_{\nu=0}^{n-1} Y^{-1}(x_0+(\nu+1)h) \cdot B(x_0+\nu h) z(x_0+\nu h) + Y(x_0+nh) \sum_{\nu=0}^{n-1} Y^{-1}(x_0+\nu h) w(x_0+\nu h)$$

where $Y(t)$ is a solution of the following equation:

$$\begin{cases} \nabla Y(x) = AY(x-h) \\ Y(x_0) = 1+A \qquad (A \neq -1). \end{cases}$$

In the above lemma ∇ denotes the back-ward difference operator and using the above lemma we have the following lemma.

Lemma 1.2. *Consider the difference equation:*

$$\begin{cases} \nabla z(x_0+nh) = \rho z(x_0+(n-1)h) + B(x_0+(n-1)h)z(x_0+(n-1)h) \\ \qquad + w(x_0+(n-1)h) \qquad (\rho:\ \text{constant}) \\ z(x_0) = z_0 \qquad (|z_0| \leqq C) \end{cases}$$

and suppose that

(1) $$\sum_{\nu=0}^{\infty} |Y(x_0+\nu h)| = C_0 < \infty$$

where $Y(t)$ is a solution of the difference equation:

$$\begin{cases} \nabla Y(x_0+nh) = \rho Y(x_0+(n-1)h) \\ Y(x_0) = (1+\rho) \qquad (\rho \neq -1), \end{cases}$$

(2) $$|B(x_0+nh)| \leqq \frac{1}{4(C_0+1)} \qquad (n=0, 1, 2, \cdots),$$

and

(3) $$|w(x_0+nh)| \leqq \frac{C}{2C_0} \qquad (n=0, 1, 2, \cdots).$$

Then

$$|z(x_0+nh)| \leqq 2C \qquad (n=0, 1, 2, \cdots).$$

The difference equation (1.7) may be written in the form:

(1.8) $$\nabla e_n = \rho e_n + (hf_y(x_n+\eta_n) - \rho)e_n + E_{n+1}.$$

Hence from Lemma 1.2 we may obtain the next theorem.

Theorem 1. *Considering the difference equation* (1.8) *under the conditions*:

(1) $$\left|\frac{\partial f}{\partial y}(x, y)\right| \leqq k \qquad (k\text{: constant}),$$

and

(2) $$|E_n| \leqq \frac{C}{2C_0},$$

we have

$$|e_n| \leqq 2C$$

for

$$0<h<\frac{1}{k}\left(\rho+\frac{1}{4\sum_{\nu=0}^{\infty}|Y(x_0+\nu h)|}\right) \qquad \left(k \neq 0,\ -1<\rho<-\frac{3}{4}\right).$$

Next we shall show that under certain conditions the solution of difference equation (1.8) tends to zero as $n \to \infty$. Before giving Theorem 2 we shall present a lemma.

Lemma 1.3. *Consider the difference equation*:

$$\begin{cases} \nabla z(x_0+nh)=Az(x_0+(n-1)h)+B(x_0+(n-1)h)z(x_0+(n-1)h) \\ \qquad\qquad +w(x_0+(n-1)h) \\ z(x_0)=z_0 \end{cases}$$

under the following conditions:

(1) *the solution of the difference equation*:

$$\begin{cases} \nabla Y(x_0+nh)=AY(x_0+(n-1)h) \\ Y(x_0)=1 \end{cases}$$

tends to zero as $n \to \infty$,

(2) $$|B(x_0+nh)| \leqq \frac{1}{4}e^{(2\lambda-\tilde{\lambda})} \qquad (n=0, 1, 2, \cdots)$$

(3) $$|w(x_0+nh)| \leqq ae^{-\lambda_1(x_0+nh)}$$

where

$$\lambda>-\log(1+A),\ \lambda>\tilde{\lambda}>0,\ \lambda_1>\frac{\lambda}{h} \quad \text{and} \quad \frac{e^{-\lambda_1 x_0}}{\sum_{\nu=0}^{\infty} e^{-(\lambda_1 h-\lambda)\nu}}>a>0,$$

then

$$|e_n| \leqq 2e^{-\tilde{\lambda}n} \qquad (n=0, 1, 2, \cdots).$$

Consequently we have the next theorem.

Theorem 2. *If we choose*

$$0<h<\frac{1}{k}\left(A+\frac{1}{4}e^{\lambda}(e^{(\lambda-\tilde{\lambda})}-1)\right),$$

where the constants A, λ, λ_1 *are given in Lemma* 1.3 *and the constant* k *is given in the following condition* (1), *then the error* e_n *obtained from* (1.8) *satisfies the inequality*

$$|e_n| \leqq 2e^{-\tilde{\lambda}n} \qquad (n=0, 1, 2, \cdots)$$

under the following conditions:

(1) $$\left|\frac{\partial f}{\partial y}(x, y)\right| \leqq k \qquad (k:\ constant)$$

(2) $$|E(x_0+nh)| \leqq ae^{-\lambda_1(x_0+nh)}$$

where the constants $a, \tilde{\lambda}, \lambda_1$ *are given in Lemma* 1.3.

§2. In §1 we consider the propagation of error of a special approximation method. And it will be investigated in this section the propagation of error of general one step methods. General one step method may be written in the form with an appropriate function $\Phi(x, y: h)$, using the same notation as in **§1**,

(2.1) $$y_{n+1} = y_n + h\Phi(x_n, y_n: h) - T_{n+1}$$

and

(2.2) $$y(x_{n+1}) = y(x_n) + h\Phi(x_n, y(x_n): h) + R_{n+1}$$

where

$$\Phi(x, y: h) = \begin{cases} \dfrac{z(x+h)-z(x)}{h} & (h \neq 0) \\ f(x, y) & (h=0) \end{cases}$$

and the function $z(x)$ is a solution of (1.1).

From the equations (2.1) and (2.2), we may derive the equation:

(2.3) $$\begin{aligned} e_{n+1} &= e_n + h\{\Phi_y(x_n, \eta_n: h)\} + T_{n+1} + R_{n+1} \\ &= e_n + h\Phi_y(x_n, \eta_n: h) + E_{n+1}. \end{aligned}$$

And corresponding to Theorem 1 we have the next result.

Theorem 3. *Considering the difference equation* (2.3) *under the conditions*

(1) $$|e_0| \leqq |1+A| \qquad \left(-1 < A < \frac{-1}{2}\right)$$

(2) $$\left|\frac{\partial \Phi}{\partial y}(x, y)\right| \leqq k \qquad (k:\ constant)$$

(3) $$|E_n| \leqq \frac{1}{2(C+1)}$$

where

$$C = -\left(1 + \frac{1}{A}\right),$$

we have

$$|e_n| \leqq 2C$$

for

$$0 < h < \frac{1}{k}\left(A + \frac{1}{2C}\right).$$

In **§1** and **§2** we investigated the propagation of error of open formula. Then, using the same idea, we may investigate the propagation of error of closed formula and of general n-setp methods.

Detailed proofs and related results will appear elsewhere.

References

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