108. On Exponential Semigroups. II

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(Comm. by Kenjiro SHODA, M. J. A., Sept. 12, 1972)

1. Introduction. Tamura and Shafer proved in [3] the following:

Theorem 1. *If S is an exponential archimedean semigroup with idempotent, then S is an ideal extension of I by N where I is the direct product of an abelian group G and a rectangular band B and N is an exponential nil-semigroup.*

However, the converse is not necessarily true. For example, let $S=\{a,b,c,d\}$ be the semigroup of order 4 defined by $(x, y \in S)$

$$xy=y \text{ for } y\neq d \text{ and all } x;\ xd=a \text{ for } x\neq c;\ cd=b.$$

S is the ideal extension of a right zero semigroup $\{a,b,c\}$ by a null semigroup of order 2. Associativity of S is easily verified, but S is not exponential:

$$(cd)^2=b^2=b, \qquad c^2d^2=ca=a.$$

The purpose of this paper is to prove Theorem 2 which characterizes exponential ideal extensions of I by N, and to give an alternate proof of the fact that I is completely simple. See the definition of the used terminology in [3] and [1]. The notation may be different from that in [1].

Theorem 2. *S is an exponential archimedean semigroup with idempotent if and only if S is an ideal extension of the direct product $I=\Lambda\times G\times M$ of a left zero semigroup Λ, an abelian group G, and a right zero semigroup M by an exponential nil-semigroup N, with product determined by three partial homomorphisms $\varphi: N\backslash\{0\}\to M$, $\mathfrak{G}: N\backslash\{0\}\to G$, $\psi: N\backslash\{0\}\to\Lambda$ in the following manner. Let (λ, a, μ), $(\nu, b, \eta)\in\Lambda\times G\times M$, $s, t\in N\backslash\{0\}$.*

$$(2.1)\qquad \begin{cases} (\lambda, a, \mu)\cdot s=(\lambda, a(s\mathfrak{G}), s\varphi) \\ s\cdot(\lambda, a, \mu)=(\psi s, (s\mathfrak{G})a, \mu) \\ (\lambda, a, \mu)\cdot(\nu, b, \eta)=(\lambda, ab, \eta) \\ s\cdot t=\begin{cases} st & \text{if } st\neq 0 \text{ in } N \\ (\psi s, (s\mathfrak{G})(t\mathfrak{G}), t\varphi) & \text{if } st=0 \text{ in } N \end{cases} \end{cases}$$

2. Alternate proof of complete simpleness of I. In [3] Anderson's theorem on bicyclic subsemigroup was used, but we will derive primitiveness of idempotent elements. Assume that S is an exponential archimedean semigroup. Let e be an idempotent element of S and let $I=SeS$. Since $I\subseteq SaS$ for all $a\in S$, I is the kernel of S and hence I is simple. Let e and f be idempotents such that $ef=fe=f$. Now IeI

$=(SeS)e(SeS)=(Se)(SeS)(eS)=(Se)(SfS)(eS)=(SeS)f(SeS)=IfI$. So there exist $x', y' \in I$ such that $x'fy'=e$. Let $x=ex'f$ and $y=fy'e$. Then $xy=(ex'f)(fy'e)=e(x'fy')e=e$. Since $y=ye, yx=(ye)x=y(xy)x=(yx)^2=y^2x^2$ by exponentiality while $xy=e$ implies $e=(xy)^2=(xy)(xy)=x(yx)y=x(y^2x^2)y=(xy)(yx)(xy)=e(yx)e=yx$ as we have $ey=y$ and $xe=x$ by $ef=fe=f$. Finally $f=ef=(yx)f=y(ex'f)f=y(ex'f)=yx=e$. Hence I is completely simple.

3. Preliminaries on ideal extension. Let D be a completely simple semigroup and let $D=\mathcal{M}(\Lambda, G, M; \boldsymbol{F})$ be the Rees regular matrix representation of D where Λ is a left zero semigroup, G a group, M a right zero semigroup and $\boldsymbol{F}$ a sandwich matrix. Each element of D is expressed as

$$(\lambda, x, \mu), \lambda \in \Lambda, x \in G, \mu \in M.$$

The following are already known in [1], [2] or will be easily proved by readers.

(3.1) Let $h: M \to M$ and $p: M \to G$ be mappings. If we define $\varphi_{(p,h)}: D \to D$ by

$$(\lambda, x, \mu)\varphi_{(p,h)}=(\lambda, x(\mu p), \mu h)$$

then $\varphi_{(p,h)}$ is a right translation of D. Every right translation of D is obtained in this manner, and the correspondence $(p, h) \to \varphi_{(p,h)}$ is one to one.

(3.2) Let $k: \Lambda \to \Lambda$ and $q: \Lambda \to G$ be mappings. If we define $\psi_{((k,q))}: D \to D$ by

$$\psi_{((k,q))}(\lambda, x, \mu)=(k\lambda, (q\lambda)x, \mu)$$

then $\psi_{((k,q))}$ is a left translation of D and every left translation of D is obtained in this manner. The correspondence $((k, q)) \to \psi_{((k,q))}$ is one to one.

(3.3) Let $\boldsymbol{F}=(f_{\mu\lambda}), \mu \in M, \lambda \in \Lambda$. Then $\varphi_{(p,h)}$ is linked with $\psi_{((k,q))}$ if and only if

$$(\mu p)\cdot f_{\mu h,\lambda}=f_{\mu,k\lambda}\cdot(q\lambda) \qquad \text{for all } \mu \in M, \lambda \in \Lambda.$$

In the present paper we deal with $I=\Lambda \times G \times M$ (for D) in which all $f_{\mu\lambda}$ equal to the identity e of G. Hence we have

(3.4) $$\mu p=q\lambda \qquad \text{for all } \mu \in M, \lambda \in \Lambda.$$

Thus p and q are constant mappings taking the same value in G. The p and q are denoted by p_a and q_a respectively, that is, $\mu p_a=a$, $q_a\lambda=a$ for all $\mu \in M$, all $\lambda \in \Lambda$.

$$\psi_{((k_1,q_a))}\cdot\psi_{((k_2,q_b))}=\psi_{((k_1\cdot k_2,q_{ab}))},$$
$$\varphi_{(p_a,h_1)}\cdot\varphi_{(p_b,h_2)}=\varphi_{(p_{ab},h_1\cdot h_2)}.$$

The translational hull $\mathcal{H}(I)$ of I consists of $(\psi_{((k,q_a))}, \varphi_{(p_a,h)})$ and

$$(\psi_{((k_1,q_a))}, \varphi_{(p_a,h_1)})(\psi_{((k_2,q_b))}, \varphi_{(p_b,h_2)})$$
$$=(\psi_{((k_1\cdot k_2,q_{ab}))}, \varphi_{(p_{ab},h_1\cdot h_2)}).$$

(3.5) $\varphi_{(p_a,h)}$ $(\psi_{((k,q_a))})$ is an inner right (left) translation of I if and only if $h\,(k)$ is a constant mapping. We redenote $h\,(k)$ by $h_{\mu_0}\,(k_{\lambda_0})$, i.e. $\mu h_{\mu_0} = \mu_0$ for all $\mu \in M$, ($k_{\lambda_0}\lambda = \lambda_0$ for all $\lambda \in \Lambda$). Then $(\lambda, x, \mu)\varphi_{(p_a, h_{\mu_0})} = (\lambda, x, \mu)(\xi, a, \mu_0)$ for all $\xi \in \Lambda$.

$$\psi_{((k_{\lambda_0}, q_a))}(\lambda, x, \mu) = (\lambda_0, a, \eta)(\lambda, x, \mu) \qquad \text{for all } \eta \in M.$$

The translational hull $\mathcal{H}(I)$ is isomorphic onto the direct product $\mathcal{T}_\Lambda \times G \times \mathcal{T}_M = \{[k, a, h] : k \in \mathcal{T}, a \in G, h \in \mathcal{T}_M\}$ where $\mathcal{T}_\Lambda$ and $\mathcal{T}_M$ are the full-transformation semigroups on Λ and M respectively, under the map

$$(\psi_{((k, q_a))}, \varphi_{(p_a, h)}) \mapsto [k, a, h].$$

Let $\mathcal{D}(I) = \{(\psi_{((k_{\lambda_0}, q_a))}, \varphi_{(p_a, h_{\mu_0})}) : \lambda_0 \in \Lambda, a \in G, \mu_0 \in M\}$.
Since I is weakly reductive, $\mathcal{D}(I)$ is isomorphic onto I under the composition:

$$(\psi_{((k_{\lambda_0}, q_a))}, \varphi_{(p_a, h_{\mu_0})}) \mapsto [k_{\lambda_0}, a, h_{\mu_0}] \mapsto (\lambda_0, a, \mu_0).$$

After identifying, let $(\psi_{((k, q_a))}, \varphi_{(p_a, h)}) = [k, a, h]$.

4. Exponential ideal extension. Since I is weakly reductive, an ideal extension of I by N is determined by a partial homomorphism P^* of $N^* = N \backslash \{0\}$ into $\mathcal{H}(I)$ which satisfies

$$P^*(s)P^*(t) \in \mathcal{D}(I) \qquad \text{if } s, t \in N^* \text{ and } st = 0 \text{ in } N$$

(See [1], [2]). For the notational convenience $P^*(s)$ is denoted by

$$P^*(s) = [k^{(s)}, g^{(s)}, h^{(s)}] \quad \text{where} \quad k^{(s)} \in \mathcal{T}_\Lambda, g^{(s)} \in G \quad \text{and} \quad h^{(s)} \in \mathcal{T}_M.$$

Now extend P^* to P on $S = I \cup N^*$ as follows

$$\text{(4.1)} \qquad \begin{cases} P(s) = P^*(s) & \text{if } s \in N^* \\ P(\lambda, a, \mu) = [k_\lambda, a, h_\mu & \text{if } (\lambda, a, \mu) \in I \end{cases}$$

where k_λ and h_μ are constant mappings. After identifying $[k_\lambda, a, h_\mu]$ with (λ, a, μ), the operation on S can be expressed as follows:

$$\text{(4.2)} \qquad \begin{cases} (\lambda, a, \mu)(\nu, b, \eta) = (\lambda, ab, \eta) = P(\lambda, a, \mu)P(\nu, b, \eta) \\ (\lambda, a, \mu) \cdot s = (\lambda, a \cdot g^{(s)}, \mu h^{(s)}) = P(\lambda, a, \mu)P^*(s) \\ s(\lambda, a, \mu) = (k^{(s)}\lambda, g^{(s)}a, \mu) = P^*(s)P(\lambda, a, \mu) \\ s \cdot t = \begin{cases} st & \text{if } st \neq 0 \text{ in } N \\ P^*(s)P^*(t) = (\lambda_0, a_0, \mu_0) & \text{(if } st = 0 \text{ in } N, (\lambda_0, a_0, \mu_0) \\ & \text{is uniquely determined.)} \end{cases} \end{cases}$$

Accordingly

$$\text{(4.3)} \qquad \begin{cases} xy = P(x)P(y) = P(xy) & \text{if } xy \in I \\ P(xy) = P(x)P(y) & \text{for all } x, y \in S. \end{cases}$$

Thus P is a homomorphism of S into $\mathcal{D}(I)$.

Assume that we obtain an exponential ideal extension S of I by an exponential nil-semigroup N. Let $s \in N^*$. Since N is nil, there is a positive integer n such that $s^n \in I$, hence $(P(s))^n \in \mathcal{D}(I)$, i.e. $k^{(s)n}$ and $h^{(s)n}$ are constant mappings. Let $(P(s))^n = [k_{\lambda_1}, g_1, h_{\mu_1}]$.
By exponentiality of S,

$$((\lambda, a, \mu) \cdot s)^n = (\lambda, a, \mu)^n s^n \qquad \text{for all } (\lambda, a, \mu) \in I, s \in N^*.$$

By (4.3) we get

$$([k_\lambda, a, h_\mu][k^{(s)}, g^{(s)}, h^{(s)}])^n = [k_\lambda, ag^{(s)}, h_\mu h^{(s)}]^n \\ = [k_\lambda, (ag^{(s)})^n, h_\mu h^{(s)}]. \tag{4.4}$$

On the other hand

$$[k_\lambda, a, h_\mu]^n[k^{(s)}, g^{(s)}, h^{(s)}]^n = [k_\lambda, a^n, h_\mu][k_{\lambda_1}, g_1, h_{\mu_1}] \\ = [k_\lambda, a^n g_1, h_{\mu_1}]. \tag{4.5}$$

From the equality "(4.4)=(4.5)", we have

$$h_\mu h^{(s)} = h_{\mu_1} \qquad \text{for all } \mu \in M,$$

that is, $\mu h^{(s)} = \mu_1$ for all $\mu \in M$.

Hence $h^{(s)}$ is a constant mapping. Similarly, starting $(s\cdot(\lambda, a, \mu))^n = s^n(\lambda, a, \mu)^n$, we can prove that $k^{(s)}$ is a constant mapping.

Consequently P^* induces mappings

$$\psi : N^* \to \Lambda, \quad \mathfrak{G} : N^* \to G, \quad \varphi : N^* \to M$$

such that $P^*(s) = [k_{\psi s}, s\mathfrak{G}, h_{s\varphi}]$. P^* is a partial homomorphism of N^* into $\mathcal{D}(I)$, and hence P is a homomorphism S into $\mathcal{D}(I)$. Thus we have obtained (2.1).

Conversely assume that $\psi, \mathfrak{G}, \varphi$ are given and that S is defined by (2.1). The three mappings induce a partial homomorphism P^* of N^*, $P^*(s) = [k_{\psi s}, s\mathfrak{G}, h_{s\varphi}]$, and hence induces a homomorphism P of S into $\mathcal{D}(I)$ by (4.1). Associativity of S is assured by the general theory of ideal extension of a weakly reductive semigroup, and so we need only to show exponentiality of S:

$$(xy)^m = x^m y^m \qquad \text{for all } x, y \in S, \text{ for } m > 1. \tag{4.6}$$

First note that I is medial; hence $P(S)$ is medial.

If $x^m y^m \notin I$ then (4.6) is obtained by the exponentiality of N. If $x^m y^m \in I$, then $(xy)^m \in I$ and, by (4.3) and the above remark,

$$x^m y^m = P(x^m)P(y^m) = (P(x))^m (P(y))^m,$$
$$(xy)^m = P(xy)(P((xy)^{m-1})) = P(xy)(P(xy))^{m-1} = P(x)P(y)(P(x)P(y))^{m-1}$$
$$= P(x)P(y)(P(x))^{m-1}(P(y))^{m-1} = P(x)(P(x))^{m-1}P(y)(P(y))^{m-1}$$
$$= (P(x))^m (P(y))^m.$$

Hence (4.6) has been proved.

An ideal extension of I by N determined by a partial homomorphism $N^* \to \mathcal{D}(I)$ is called a strict ideal extension.

Thus we have Theorem 2′ which is a restatement of Theorem 2 and also describes the "medial" case. The medial case is an immediate consequence from the fact that $P(S)$ is medial.

Theorem 2′. *S is an exponential (medial) archimedean semigroup with idempotent if and only if S is a strict ideal extension of the direct product of an abelian group G and a rectangular band B by an exponential (medial) nil-semigroup N.*

Finally we exhibit an example of exponential semigroup which is not medial. It is sufficient to show such a nil-semigroup. Let F be the free semigroup generated by two letters a, b and let S^* be a subset of F defined by

$$S^* = \{a, b, ab, a^2, ba, a^2b, aba, a^2ba\}$$

and $$I = F \backslash S^*.$$

Then I is an ideal of F. Let $S = F/I$. S is an exponential semigroup of order 9 which is not medial since $a^2ba \neq aba^2 = 0$.

References

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