106. *Modified Korteweg - de Vries Equation and Scattering Theory*

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1. Introduction. Gardner, Greene, Kruskal and Miura (G. G. K. M.) [1] have discovered that the initial value problem for the Korteweg - de Vries (KdV) equation

$$v_t + 6vv_x + v_{xxx} = 0$$

(subscripts x and t denoting partial differentiations) may be exactly solved by the direct and inverse scattering theory of the one dimensional Schrödinger operator. Zakharov and Shabat [9] have then developed an analogue of G. G. K. M. theory for the non-linear Schrödinger equation

$$iu_t + 2^{-1}u_{xx} + |u|^2 u = 0 \tag{1}$$

relating it to the scattering theory of the differential operator

$$L_u = i\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}D - i\begin{bmatrix}0 & u\\ u^* & 0\end{bmatrix} \qquad D = d/dx$$

with complex potential u (u^* being its complex conjugate).

Recently Wadati [8] and the present author [7] have noted that the modified KdV equation

$$v_t + 6v^2 v_x + v_{xxx} = 0 \tag{2}$$

(v being real-valued) can be also related to the operator L_u. In [7] a family of particular solutions of (2) have been explicitly constructed based on this relation. In this paper we supplement [7] with the description of more general aspect of the theory.

2. Evolution equations for linear operators. Lax [3], [4] has rewritten the KdV equation into the evolution equation for the Scrödinger operator. An analogous result also holds for equation (2): Put

$$A_v = -4D^3 + 3\begin{bmatrix}-v^2 & iv_x\\ iv_x & -v^2\end{bmatrix}D + 3D\begin{bmatrix}-v^2 & iv_x\\ iv_x & -v^2\end{bmatrix}$$

where v is a real valued function. Then by direct calculation, the equation (2) is rewritten into the form

$$(L_{iv})_t = [A_v, L_{iv}] = A_v L_{iv} - L_{iv} A_v. \tag{3}$$

This expression has been obtained in [7].

Remark 1. Put

$$B_u = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}(D^2 + 2^{-1}|u|^2) - 2^{-1}\begin{bmatrix}0 & u\\ u^* & 0\end{bmatrix}D - 2^{-1}D\begin{bmatrix}0 & u\\ u^* & 0\end{bmatrix}$$

and

$$B'_u=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}(D^2-2^{-1}|u|^2)-2^{-1}i\begin{bmatrix}0 & u\\ u^* & 0\end{bmatrix}D-2^{-1}iD\begin{bmatrix}0 & u\\ u^* & 0\end{bmatrix}$$

Then (1) is written as

$$(L_u)_t=i[B_u, L_u]$$

and the argument of section 4 works also for (1). The equation

$$iu_t+2^{-1}u_{xx}-|u|^2u=0$$

is written as

$$(L'_u)_t=i[B'_u, L'_u]$$

where

$$L'_u=i\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}D+\begin{bmatrix}0 & u\\ u^* & 0\end{bmatrix}.$$

L'_u is essentially one dimensional Dirac operator.

Remark 2. In [9], (1) has been written as an evolution equation for the operator

$$M_u=i\begin{bmatrix}1+p & 0\\ 0 & 1-p\end{bmatrix}D+\begin{bmatrix}0 & u^*\\ u & 0\end{bmatrix}$$

(p being a real constant). Wadati [8] has written (2) as an evolution equation for M_{iv}.

3. Jost function and the scattering data. We follow [9] for the generality of scattering theory of L_u. Consider the eigenvalue problem

(4) $$L_u y=\zeta y \qquad y={}^t(y_1, y_2)\ .$$

Then if y is a solution of (4), $y^\#={}^t(y_2^*, -y_1^*)$ is a solution of (4), ζ being replaced by ζ^*. If u is integrable, one can show that for each $\zeta=\xi+i\eta$, $\eta\geq 0$, there exist unique solutions (called Jost functions) ϕ and ψ of (4) which behave as ${}^t(1,0)\exp(-i\zeta x)$, $x\to-\infty$, and ${}^t(0,1)\exp(i\zeta x)$, $x\to\infty$, respectively. ϕ and ψ are analytic in ζ, $\mathrm{Im}\,\zeta>0$. If $\zeta=\xi$ real, then ψ and $\psi^\#$ are independent solutions of (4). So one can express ϕ as

(5) $$\phi=a(\xi)\psi^\#+b(\xi)\psi.$$

We have $a(\xi)=\det(\phi, \psi)$ and the function $a(\xi)$ can be extended to the analytic function $a(\zeta)$, $\mathrm{Im}\,\zeta>0$. Shabat [5] showed that under the additional integrability condition on u, one can express $a(\zeta)$ as

$$a(\zeta)=1+\int_0^\infty f(t)\exp(i\zeta t)dt$$

for some f in $L^1(0,\infty)$. If moreover $U(x)\exp(\varepsilon|x|)$ is integrable for some $\varepsilon>0$, then $a(\zeta)$ has only finite number of zeros in $\mathrm{Im}\,\zeta>0$. Suppose further that all of zeros in $\mathrm{Im}\,\zeta>0$ of $a(\zeta)$ are simple and denote them by $\zeta_1, \cdots, \zeta_N$. For $\zeta=\zeta_j$, Jost functions are linearly dependent:

(6) $$\phi(x,\zeta_j)=d_j\psi(x,\zeta_j).$$

By the asymptotic property, they are square-integrable. We have

(7) $$a'(\zeta_j)=-2id_j\int_{-\infty}^\infty \psi_1\psi_2(x,\zeta_j)dx.$$

Put $c_j=d_j/a'(\zeta_j)$. The functions $a(\zeta), b(\xi)$ and the numbers $c_1, \cdots, c_N$

are called the scattering data of the operator L_u.

Suppose that $u=iv$, purely imaginary. Then

(8) $\phi(-\zeta^*) = {}^t(\phi_1^*(\zeta), -\phi_2^*(\zeta)) \qquad \psi(-\zeta^*) = {}^t(-\psi_1^*(\zeta), \psi_2^*(\zeta))$.

So we have $a^*(\zeta)=a(-\zeta^*)$. Let M be a non-negative integer such that $2M \leq N$. Let σ be the permutation among natural numbers between 1 and N defined by $\sigma(j)=j+1$, j odd $\leq 2M$; $\sigma(j)=j-1$, j even $\leq 2M$; $\sigma(j)=j$, $j>2M$. Then $\zeta_{\sigma(j)}=-\zeta_j^*$. By (7) and (8), we have $c_{\sigma(j)}=c_j^*$. It is also easy to show that $b^*(\xi)=-b(-\xi)$. Converse statement will be formulated in section 5 under the assumption that $b(\xi)\equiv 0$.

4. Time variation of the scattering data. Let us now suppose that smooth real-valued function $v=v(t)=v(x,t)$ is a solution of (2) which is rapidly decreasing in x for each t. We shall derive the time dependence of the scattering data of $L_{iv(t)}$. In this section corresponding Jost functions and scattering data contain the additional variable t.

We differentiate the relation $L_{iv}\phi=\zeta\phi$ with respect to t. Making use of (3), we see that $\phi_t - A_v\phi$ again satisfies (4). Because it behaves like $4i\zeta^3 \cdot {}^t(1,0) \exp(-i\zeta x)$ as $x \to -\infty$, by the uniqueness of Jost functions, we have the differential equation which show the time variation of the Jost function:

$$\phi_t - A_v\phi = 4i\zeta^3\phi.$$

Putting (5) into this equation for $\zeta=\xi$ and then eliminating ψ_t and $\psi_t^\#$ by the similar differential equations, we get an identity

$$a_t\psi^\# + (b_t - 8i\xi^3 b)\psi = 0.$$

Thus we have

$$a(\xi,t)=a(\xi,0) \qquad b(\xi,t)=b(\xi,0)\exp(8i\xi^3 t).$$

$a(\zeta,t)$ is independent of t and so are its zeros. Differentiation of (6) with respect to t leads to

$$c_j(t)=c_j(0)\exp(8i\zeta_j^3 t).$$

5. Construction of generalized soliton solutions. As in [1] and [9], application of the inverse scattering theory leads to the construction of a family of particular solutions including the soliton solutions as the simplest case.

Let σ be the permutation defined in section 3 and ζ_j, $c_j(0)$ $(1\leq j\leq N)$ satisfy the conditions formulated there with respect to σ. Put

$$c_j=c_j(t)=c_j(0)\exp(8i\zeta_j^3 t) \qquad \lambda_j=\lambda_j(x,t)=c_j(t)^{1/2}\exp(i\zeta_j x).$$

Theorem. *Let $\psi_{1j}(x,t)$ and $\psi_{2j}(x,t)$ be the solution of the system of $2N$ linear algebraic equations*

$$(9) \qquad \begin{aligned} \psi_{1j} + \Sigma_k \lambda_j\lambda_k^*(\zeta_j-\zeta_k^*)^{-1}\psi_{2k}^* &= 0 \\ -\Sigma_k \lambda_k\lambda_j^*(\zeta_j^*-\zeta_k)^{-1}\psi_{1k} + \psi_{2j}^* &= \lambda_j^*. \end{aligned} \qquad (1\leq j\leq N)$$

Then

$$u(x,t) = -2i\Sigma_j \lambda_j^*(x,t)\psi_{2j}^*(x,t)$$

is purely imaginary and $v(x,t)=i^{-1}u(x,t)$ is a solution of (2).

Proof of this theorem is similar to that of [6] where analogous result for the KdV equation has been described. We show first that the coefficient matrix of (9) is non-degenerate. The relations $L_{iv}\psi_j = \zeta_j\psi_j$ for $\psi_j = {}^t(\psi_{1j}, \psi_{2j})$ are then derived (See Kay and Moses [2] where analogous results for the Schrödinger operator are proved). The formulas for the derivatives of v, for example

$$v_t = 16i\Sigma_j(-\zeta_j^3\psi_{1j}^2 + \zeta_j^{*3}\psi_{2j}^{*2}) ,$$

are then obtained. See [7] for the detail. The system of equations (9) has been obtained in [9] where analogous construction for the equation (1) has been discussed. We can show that the functions constructed there in fact satisfy the equation (1) by the method described here.

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