101. *On Complex Parallelisable Manifolds and their Small Deformations*

By Iku NAKAMURA

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0°. Introduction. By a complex parallelisable manifold we mean a compact complex manifold with the trivial holomorphic tangent bundle. Wang [7] showed that a complex parallelisable manifold is the quotient space of a simply connected complex Lie group by one of its discreet subgroups.

This note is a preliminary report on our recent results on complex parallelisable manifolds and their small deformations. Details will appear in the forthcoming paper [5].

1°. Let X be a compact complex manifold of $\dim n$. We denote by $\mathcal{O}$ and Ω^p the sheaf of germs of holomorphic functions and the sheaf of germs of holomorphic p-forms. We define $h^{p,q}=\dim H^q(X,\Omega^p)$, $P_m=\dim H^\circ(X,(\Omega^n)^{\otimes m})$, $r=$ the number of linearly independent closed holomorphic 1-forms, $\kappa=$Kodaira dimension of X and $b_i=$the i-th Betti number of X. S. Iitaka proposed the problem whether all P_m and κ are deformation-invariants ([1]).

2°. Proposition. *A simply connected complex Lie group G of $\dim_C n$ is analytically homeomorphic to C^n as a complex manifold.*

Proof. We shall prove the proposition by induction on n. It is obvious in case of $n=1$. Let the Lie group be G. If $n\geqq 2$, we can take a connected normal subgroup N. Then the canonical mapping $\pi: G\to G/N$ defines a holomorphic fiber bundle. Since both G/N and N are connected and simply connected we obtain the proposition by the induction hypothesis and Grauert's theorem.

3°. We define a complex parallelisable manifold to be solvable if the corresponding Lie group is solvable. From now on we assume X to be solvable. Note that the universal covering of X is analytically homeomorphic to C^n by the above Proposition.

Theorem 1. *Three dimensional solvable manifolds are classified into the following four classes.*

	Lie group	b_1	r	$h^{0,1}$	Structure (Albanese mapping)
(1)	abelian	6	3	3	complex torus
(2)	nilpotent	4	2	2	T^1-bundle over T^2
(3a)	solvable	2	1	1	T^2-bundle over T^1
(3b)	solvable	2	1	3	T^2-bundle over T^1

where T^1 and T^2 denote complex tori of dimension 1 and 2, respectively. We can calculate small deformations explicitly by solving the differential equations in the deformation theory of Kodaira-Spencer-Kuranishi. Computing numerical characters of their small deformations we obtain interesting results:

(i) $h^{p,q}((p,q)\neq(0.0))$, r, κ, and P_m are not necessarily deformation invariants.

(ii) In case of (3b) there exists a small deformation whose universal covering is not analytically homeomorphic to $\boldsymbol{C}^3$. This example is constructed as follows.

Take a unimodular (2, 2) matrix A with $\operatorname{tr} A \geqq 3$. Let J be the Jordan form of A, that is to say, $J=P^{-1}AP$ for a non-singular matrix P. Let α and $\alpha^{-1}(\alpha>1)$ be the eigenvalues of A. We note that α is real since $\operatorname{tr} A$ is greater than 2.

Let T^2 be a complex torus defined by the period matrix $(P^{-1}, \tau P^{-1})$ where τ is a complex number with a positive imaginary part. We define two automorphisms g_1, g_2 of $\boldsymbol{C}\times T^2$ as follows

$$g_1 : (z_1, z_2, z_2)\rightarrow(z_1+2\pi i, z_2, z_3)$$
$$g_2 : (z_1, z_2, z_3)\rightarrow(z_1+\beta, \alpha^{-1}z_2, \alpha z_3)$$

where $\beta=\log \alpha>0$.

Letting Γ_1 be an automorphism group of $\boldsymbol{C}\times T^2$ generated by g_1 and g_2 we have a compact complex manifold $X=\boldsymbol{C}\times T^2/\Gamma_1$. Then X is parallelisable and $h^{0,1}(X)=3$. In fact $\varphi_1=dz_1, \varphi_2=e^{z_1}dz_2$ and $\varphi_3=e^{-z_1}dz_3$ form a basis of $H^0(X, \Omega^1)$, and $\bar{\varphi}_1, e^{z_1-\bar{z}_1}\bar{\varphi}_2$ and $e^{-z_1+\bar{z}_1}\bar{\varphi}_3$ generate $H^1(X, \mathcal{O}) \cong H^{0,1}_{\bar{\partial}}(X)$ (Dolbeault isomorphism). We can also consider X to be a quotient space of $\boldsymbol{C}^3$, i.e., $X=\boldsymbol{C}^3/\Gamma$. Then an interesting small deformation X_t of X is given as follows:

$X_t=W_t/\Delta_t$ where $W_t=\{(w_1, z_2, z_3);\, w_1+t\bar{z}_2\neq 0\}$ and Δ_t is defined by,

$$w_1'=e^{-\omega_1}(w_1-t\bar{\omega}_2)$$
$$z_2'=e^{-\omega_1}(z_2+\omega_2)$$
$$z_3'=e^{\omega_1}(z_3+\omega_3),\ (\omega_1, \omega_2, \omega_3)\in\Gamma.$$

For this deformation $X_t(t\neq 0)$ we have $P_m(X_t)=0$, $h^{0,1}(X_t)=2$, $r(X_t)=0$, etc.

Remark. $X_0=X$. $W_t(t\neq 0)$ is not Stein and hence the universal covering of W_t is not Stein.

We can also classify four and five dimensional complex solvable manifolds in the same way as above.

4°. **Theorem 2** (Kodaira). *Assume X to be parallelisable such that the corresponding Lie group is nilpotent. Then we have $h^{0,1}=r$.*

Theorem 3. *For a complex solvable manifold we have $b_1=2r$.*

Theorem 4. *If a complex solvable manifold satisfies the equality $h^{0,1}=r$, then any small deformation has $\boldsymbol{C}^n$ as its universal covering.*

Remark. For most complex solvable manifolds we have $h^{0,1}=r$.

References

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