100. *On Surfaces of Class VII_0*

By Masahisa INOUE

(Comm. by Kunihiko KODAIRA, M. J. A., Sept. 12, 1972)

1. In this short note we consider the surfaces satisfying the following conditions:

(*) $b_1=1, b_2=0$; the surfaces contain no curves.

We give two kinds of examples satisfying (*), and give a theorem which determines the surfaces satisfying (*) under an additional assumption. As a result of this theorem, we give three corollaries. The first of the corollaries is proved independently by Enrico Bombieri by a similar method.

Details will be published elsewhere.

2. Let $M \in SL(3, \boldsymbol{Z})$ be a unimodular matrix, with one real and two non-real eigenvalues, $\alpha, \beta, \bar{\beta}$, where $\alpha\beta\bar{\beta}=1$ and $\alpha>1$. Let $\boldsymbol{a}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$ be a real eigenvector of α and $\boldsymbol{b}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$ an eigenvector of β.

Let G_M be the group generated by the analytic automorphisms:

$$(W, Z)\to(W+m_1a_1+m_2a_2+m_3a_3, Z+m_1b_1+m_2b_2+m_3b_3),$$
$$(m_1, m_2, m_3) \in \boldsymbol{Z}^3,$$
$$(W, Z)\to(\alpha W, \beta Z),$$

of $\boldsymbol{H}\times\boldsymbol{C}$, where $\boldsymbol{H}$ is the upper half-plane. The action of G_M on $\boldsymbol{H}\times\boldsymbol{C}$ is properly discontinuous and fixed point free. Now we define an analytic surface S_M to be $\boldsymbol{H}\times\boldsymbol{C}/G_M$. Then S_M is differentiably a 3-torus bundle over a circle, $b_1(S_M)=1$, $b_2(S_M)=0$, and S_M has the following properties.

Proposition 1.

i) *S_M contains no curves,*

ii) $\dim H^0(S_M, \Theta)=\dim H^1(S_M, \Theta)=\dim H^2(S_M, \Theta)=0.$

3. Let $N=(n_{ij}) \in SL(2, \boldsymbol{Z})$ be a unimodular matrix with two real eigenvalues, $\alpha, 1/\alpha$, where $\alpha>1$. Let

$$\boldsymbol{c}=\begin{pmatrix}c_1\\c_2\end{pmatrix} \quad \text{and} \quad \boldsymbol{a}=\begin{pmatrix}a_1\\a_2\end{pmatrix}$$

be real eigenvectors of α and $1/\alpha$, respectively. We fix an arbitrary complex number t and fix two integers, p, q, such that

$$0\le p,\ q\le|\det(N-I)|-1.$$

Let $\boldsymbol{b}=\begin{pmatrix}b_1\\b_2\end{pmatrix}$ be the solution of the following equation:

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = N \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} + \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} + (a_1c_2 - a_2c_1) \cdot \begin{pmatrix} p \\ q \end{pmatrix}$$

where

$$e_i = \frac{1}{2} n_{i1}(n_{i1}-1)a_1c_1 + \frac{1}{2} n_{i2}(n_{i2}-1)a_2c_2 + n_{i1}n_{i2}a_1c_2\,; \qquad i=1,2.$$

Let $G_{N,p,q,t}$ be the group of analytic automorphisms of $\boldsymbol{H} \times \boldsymbol{C}$ generated by

$$\begin{cases} (W,Z) \to (W+c_i, Z+a_iW+b_i), & i=1,2, \\ (W,Z) \to (\alpha W, Z+t). \end{cases}$$

The action of $G_{N,p,q,t}$ on $\boldsymbol{H} \times \boldsymbol{C}$ is properly discontinuous and fixed point free. Now we define an analytic surface $S_{N,p,q,t}$ to be $\boldsymbol{H} \times \boldsymbol{C} / G_{N,p,q,t}$. Then $S_{N,p,q,t}$ is differentiably a fibre bundle over a circle of which fibre is a circle-bundle over a 2-torus, $b_1(S_{N,p,q,t})=1$, $b_2(S_{N,p,q,t})=0$. $S_{N,p,q,t}$ has the following properties.

Proposition 2.

i) $S_{N,p,q,t}$ *contains no curves,*

ii) $\dim H^0(S_{N,p,q,t}, \Theta) = \dim H^1(S_{N,p,q,t}, \Theta) = 1$, *and* $\dim H^2(S_{N,p,q,t}, \Theta) = 0$,

iii) $\{S_{N,p,q,t}\}_{t \in \boldsymbol{C}}$ *forms a locally complete family of deformations.*

4. Now we state our main theorem. Our method of proof of this theorem is similar to K. Kodaira [1, §§ 11, 12].

Theorem. *Let S be a surface satisfying the conditions* (*). *If there exists a complex line bundle $\boldsymbol{F}$ on S such that* $\dim H^0(S, \Omega^1 \otimes \mathcal{O}(\boldsymbol{F})) \neq 0$, *then S has S_M or $S_{N,p,q,t}$ as its finite unramified covering.*

5. From the above theorem, we can derive the following corollaries.

We denote by μ_K the representation of the fundamental group into $\boldsymbol{C}^*$ which defines the canonical line bundle $\boldsymbol{K}$. Then we have:

Corollary 1. *Let S be a surface satisfying* (*). *If the representation μ_K is not real, then S has S_M as its finite unramified covering.*

We denote by $[S]$ the underlying differentiable manifold of a surface S. Then we have:

Corollary 2. *There exists on* $[S_M]$ *no complex structure other than S_M.*

Corollary 3. *Every complex structure on* $[S_{N,p,q,t}]$ *belongs to the family* $\{S_{N,p,q,t}\}_{t \in \boldsymbol{C}}$.

Reference

[1] K. Kodaira: On the structure of compact complex analytic surfaces. II, III. Amer. J. Math., **88**, 682–721 (1966); **90**, 55–83 (1968).