153. *Finitary Objects and Ultrapowers*

By Tadashi OHKUMA

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Introduction. When we deal with categories of systems with structures, we often feel it desirable to set up a notion that distinguish algebraic structures, such as ordered sets or groups, from infinitistic theories, such as topological spaces. One attempt was made in [4] for concrete categories and studied particularly in connection with ultrapowers. Here the notion of finitary objects defined in [4] for concrete categories, together with one of the theorems concerning them and ultrapowers of objects, is generalized to abstract categories. Only the definitions and the results are given below. The proofs and further details are to be referred for to a paper with the same title which will be published elsewhere, of which this is an abstract.

As for the terminology, we mostly follow Isbell [2] and the terms "*extremal monomorphisms*", "*small complete*", "*left complete*", "*locally small*", "*strict monomorphisms*" etc., are used in his sense without citing the definitions. "*The co-intersection of quotient objects*" is the dual notion of "the (representable) intersection of subobjects" in [2] and "An object *co-generates* another" is the dual statement of "An object generates another" of Grothendieck [1].

§1. Finitary objects. Let C be an abstract category and Ob (C) the collection of all objects in C.

Definition 1. Let $L=\{b_\lambda\colon X_\lambda\to B\mid \lambda\in\Lambda\}$ be a set of coterminal morphisms in C. L is said to *cover* B if there is no proper extremal monomorphism $\cdot\to B$ that factors all $b_\lambda\in L$. The set L is called *compatible with* $K=\{a_\lambda\colon X_\lambda\to A\mid \lambda\in\Lambda\}$, if there exists an $f\colon B\to A$ such that $a_\lambda=fb_\lambda$ for every $\lambda\in\Lambda$. L is called *finitely compatible with* K, if for any finite subset M of Λ, $\{b_\lambda\mid\lambda\in M\}$ is compatible with $\{a_\lambda\mid\lambda\in M\}$. A is called *finitary under* B, if for any sets $L=\{b_\lambda\colon X_\lambda\to B\mid\lambda\in\Lambda\}$ and $K=\{a_\lambda\colon X_\lambda\to A\mid\lambda\in\Lambda\}$, L is compatible with K, provided the former covers B and finitely compatible with the latter. A is called *finitary*, if it is finitary under every B in Ob (C).

It can be seen that in the category of groups or of ordered sets, or in more general, in that of models of an algebraic theory, of which all primitive relations are finitary, every object is finitary, as is the intension of Definition 1, while, in the category of Hausdorff spaces, even a finite set, save for the singleton space, is not finitary.

As for the properties of finitary objects, we got the following theorems.

Theorem 1. *In a category C which has equalizers, finitary objects are closed in C under left limits of small diagrams.*

Theorem 2. *If C has pullbacks and pushouts, then an extremal subobject of a finitary object is finitary.*

For the next theorem, in case that a smaller procedure is preferable to the co-intersection of a large family of quotient objects, we prepare the notion of the image decomposition of a morphism. A decomposition $f=gh$ of a morphism f is called the *image-decomposition* of f, if h is an epimorphism and g is an extremal monomorphism. If the category has pushouts, the image-decomposition, if exists, is unique for f. If every morphism in C admits its image-decomposition, we say that the *category has image-decompositions.*

Theorem 3. *If either* (i) *the category C has pullbacks, pushouts and co-intersections of quotient objects of any object, or* (ii) *C is locally small and has pullbacks, pushouts, direct products and image-decompositions, then an object A in C is finitary in C, provided A is finitary in the full subcategory F of C generated by all objects which are cogenerated by A.*

Corollary. *In a left complete locally small category C which has pushouts, if an object is finitary in the full left closure of $\{A\}$, then it is finitary in C.*

Thus being finitary is, in a way, an intrinsic property of objects, in the sense that, if an object is finitary once in a category C with proper completeness, then it is also finitary in any extension of C.

§2. Ultrapowers. Let Ξ be a set and A_ξ an object assigned to each $\xi \in \Xi$. The canonical projection from the direct product $\prod_{\xi\in\Xi} A_\xi$ to its component A_ξ will be denoted by π_ξ^Ξ. When $\Xi' \subset \Xi$, a morphism $\pi: \prod_{\xi\in\Xi} A_\xi \to \prod_{\xi\in\Xi'} A_\xi$ is determined so that $\pi_\xi^\Xi = \pi_\xi^{\Xi'}\pi$ for every $\xi \in \Xi'$. This π is also called the projection and denoted by $\pi_{\Xi'}^\Xi$.

The notion of ultraproducts in model theory was generalized in terms of categories in [4] as follows:

Definition 2. Let Γ be a set, Φ a filter over Γ (cf. [3]) and A_ξ an object in C assigned to each $\xi \in \Gamma$. The diagram in C which consists of all $\prod_{\xi\in\Xi} A_\xi$ with $\Xi \in \Phi$ as objects and all $\pi_{\Xi'}^\Xi: \prod_{\xi\in\Xi} A_\xi \to \prod_{\xi\in\Xi'} A_\xi$ with $\Xi, \Xi' \in \Phi$ and $\Xi' \subset \Xi$ as morphisms is called a *product system relative to Φ.* The right limit, if exists, of the product system is called the *reduced product* of the family $\{A_\xi \mid \xi \in \Gamma\}$ *relative to Φ*, and denoted by $\prod_{\xi\in\Gamma} A_\xi/\Phi$. When $A_\xi = A$ for all $\xi \in \Gamma$, it is called the *reduced power* of A *relative to Φ*, and denoted by A^Γ/Φ. If Φ is maximal, $\prod_{\xi\in\Gamma} A_\xi/\Phi$ and A^Γ/Φ are respectively called an *ultraproduct* and an *ultrapower* *relative to* Φ. The canonical injection $\prod_{\xi\in\Xi} A_\xi \to \prod_{\xi\in\Gamma} A_\xi/\Phi$ is also denoted by π^Ξ.

For a direct power A^Ξ there is the so-called diagonal morphism $A\to A^\Xi$, which is denoted by d_Ξ, such that $\pi_\xi^\Xi d_\Xi : A\to A_\xi$ is the identity for every $\xi\in\Xi$. Obviously $d_{\Xi'}=\pi_{\Xi'}^\Xi d_\Xi$ for $\Xi'\subset\Xi$, and hence we have a morphism $d: A\to A^\Gamma/\Phi$ such that $d=\pi^\Xi d_\Xi$ for all $\Xi\in\Phi$. d is also called the *diagonal morphism* to the reduced power.

Those are natural extension of usual ultraproducts in model theory and the concomitant notions (cf. [3]). However, the application of the definition above to general structures sometimes brings about pathological phenomena, and many of important properties of ultraproducts are no more retained. For example, it was shown in [4] that an ultraproduct of Hausdorff spaces is always reduced to a singleton set except when the filter Φ is principal. Thus the diagonal morphism d is no more an extremal monomorphism, while in model theory d must be elementary, not to speak of its having to be an embedding.

Here one finds the essential rôle of finitary objects in the application of the definition above.

Theorem 4. *In a category C which is locally small and small complete to the both sides, if the diagonal morphism* $d: A\to A^\Gamma/\Phi$ *is an extremal monomorphism for any set* Γ *and a filter* Φ *over it, then A is finitary.*

However, the converse of this theorem does not look true, and the characterization for A so that d be always an extremal monomorphism is still open.

Here we have one comment. By replacing the term "extremal monomorphisms" in Definition 1 with general "monomorphisms", we obtain the notions of, say "*covering in the wider sense*" and "*wide-finitary*" respectively in places of "covering" and "finitary". Then for Theorem 4, we obtain the following theorem which is proved also valid: *Under the same conditions for C as in Theorem* 4, *if* $d: A\to A^\Gamma/\Phi$ *is always a monomorphism for any* Γ *and* Φ, *then A is wide-finitary.* Also it can be seen that under proper modifications and substitutions of terms, we have theorems for wide-finitary objects similar to theorems in section 1, which remain also valid. This situation is all the same for the notion of, say, "*strict-finitary*" obtained in place of "finitary" by replacing the term "extremal monomorphisms" in Definition 1 with "strict monomorphisms" (cf. [2]). There may be interesting pecuriarities for each notion. However, among those various "finitarities" the one given in Definition 1 seems most essential and we miss good examples of other modified finitarities.

References

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