152. A Treatment of Some Function Spaces used for the Study of Hypoellipticity. I

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Introduction. The space $\mathfrak{F}(\Omega)\equiv\bigcap_{i\in I}B^{\text{loc}}_{p_i,k_i}(\Omega)$ according to L. Hörmander [1] p. 45, p. 77 rather shows a common structure of the spaces belonging to a family. Then we will show here the above structure (with the extended form) described in the form of ranked space ([2] p. 4) in Theorem I-2 etc., §1, and show the concrete meaning of transcendental ranks appearing in our ranked space in Example I-1, §1. Next, we will show the concrete spaces as the special form of "the space in §1" in Theorems I-3, I-4, §2. Our extension is based on the unified description of the theorems on hypoellipticity which is related to C^∞ and related to a set of analytic functions defined in [3] p. 820 (cf. [1] p. 102, p. 178). The contents of this paper is a part of our further aim "the constructive systematization (i.e. ranked systematization by using transcendental ranks) for the theory of partial differential equation", because ranked space has a sort of totally ordered structure defined by the inclusion of pre-neighbourhoods with larger ranks.

§1. Extension of $\mathfrak{F}(\Omega)$ as a ranked space. Hereafter, we use the following notations; $K\equiv\{k(\xi)\,;\,0\leqslant k(\xi+\eta)\leqslant(1+C|\xi|)^N k(\eta)$, where $C,N>0$, $\xi,\eta\in R^n\}$, $B_{p,k}\equiv\Big\{u\,;\,u\in(\mathfrak{D}')$, $\hat u\equiv\mathfrak{F}u\to$a function, $\|u\|_{p,k}\equiv\Big((2\pi)^{-n}\int|k(\xi)\hat u(\xi)|^p\,d\xi\Big)^{1/p}<+\infty\Big\}$, where $k\in K$, $1\leqslant p\leqslant\infty$ and $\|u\|_{\infty,k}\equiv\text{ess sup}\,|k(\xi)\hat u(\xi)|$. $B^*_{p,k}\equiv\Big\{u\,;\,u\in(\mathfrak{D}')$, $\mathfrak{F}^{-1}(k(\xi)\hat u(\xi))\to$a function, $\|u\|^*_{p,k}\equiv\Big((2\pi)^{-n}\int|\mathfrak{F}^{-1}(k(\xi)\hat u(\xi))|^{p'}\,d\xi\Big)^{1/p'}<+\infty\Big\}$, where $p'=p/(p-1)$, $p'=1$ for $p=\infty$, and $p'=\infty$ for $p=1$. Ω; open connected set in R^n. $L(\Omega)\subseteq\{f\,;$ Carrier $f\subset\Omega\}$. P; diff. op. etc., $B^{\text{loc}}_{p,k}(\Omega\,;L,P)\equiv\{u\,;Pu\in(\mathfrak{D}'_\Omega),\ \varphi Pu\in B_{p,k}$ for $\forall\varphi\in L(\Omega)\}$, $B^{\text{loc}}_{p,k}(\Omega\,;L,P)\equiv\{u\,;Pu\in(\mathfrak{D}'_\Omega),\ \varphi Pu\in B^*_{p,k}$ for $\forall\varphi\in L(\Omega)\}$, $B^{\text{loc}*}_{p,k}(\Omega)\equiv B^{\text{loc}}_{p,k}(\Omega\,;C_0^\infty,1)$. If $B^{\text{loc}}_{p,k}(\Omega\,;L,P)=B^{\text{loc}}_{p,k}(\Omega\,;\tilde L,P)$ or $B^{\text{loc}*}_{p,k}(\Omega\,;L,P)=B^{\text{loc}*}_{p,k}(\Omega\,;\tilde L,P)$, we say that these spaces (in the left hand side) are countably local, where $\tilde L(\Omega)=$countable subset of $L(\Omega)$. There exists $\tilde C_0^\infty(\Omega)$ for $C_0^\infty(\Omega)$ (cf. [1] p. 44).

Definition I-1. Let I be a totally ordered set of limit or isolated ordinal numbers smaller than an inaccessible number, and let $\Gamma(n)$ be a monotone increasing function from $\{1, 2, \cdots\}$ into I (not necessarily strict). Let $B^{\text{loc}}_{pi,k_i}(\Omega; L_i, P_i)$ and $B^{\text{loc}\,*}_{pi,k_i}(\Omega; L_i, P_i)$ be countably local by $\tilde{L}_i(\Omega)=\{\varphi_{\nu,i}; \nu=1, 2, \cdots\}\subseteq L_i(\Omega)$ for any $i\in I$. (This condition "countably local" can be omitted.) Let Q_i denote $[p_i, k_i, \tilde{L}_i(\Omega), P_i]$ $(i\in I)$, and let $Q\equiv\{Q_i; i\in I\}$.

(i) Let $\hat{\Phi}_Q\equiv\bigcap_{i\in I} B^{\text{loc}}_{pi,k_i}(\Omega; L_i, P_i)$ (as a set), let $\Phi^{(l)}_Q=\bigcap_{i\in I, i\leqslant l} B^{\text{loc}}_{pi,k_i}(\Omega; L_i, P_i)$, where $l\in I$, let $B^{(l)}_Q$ be the set in $\Phi^{(l+1)}_Q$ satisfying $\overline{B^{(l)}_Q}=\Phi^{(l)}_Q$ by the topology in $\Phi^{(l)}_Q$, and let ε be a positive rational number's double sequence $\{\varepsilon_{\nu,i}\}$. Let $\hat{\mathfrak{V}}^{(l)}_{[Q,\Gamma]}\equiv[\hat{U}_l(u_0; Q, \Gamma, \varepsilon)\equiv\{u; P_i u\in\hat{\Phi}_Q, \|\varphi_{\nu,i}P_i(u-u_0)\|_{pi,k_i}\leqslant\varepsilon_{\nu,i}$ for any $i\leqslant l, \Gamma(\nu)\leqslant l\}; u_0(\in B^{(l)}_Q), \varepsilon]$. $\hat{F}_R[Q, \Gamma]$ denotes the pair $(\hat{\Phi}_Q, \{\hat{\mathfrak{V}}^{(l)}_{[Q,\Gamma]}; l\in I\})$. Since $\hat{U}_l(u_0; Q, \Gamma, \varepsilon)$ is a pre-neighbourhood, $u_0\notin\hat{U}_l(u; Q, \Gamma, \varepsilon)$ may happen. By the same way we can define $\hat{F}^*_{R[Q,\Gamma]}\equiv(\hat{\Phi}^*_Q, \{\hat{\mathfrak{V}}^{(l)*}_{[Q,\Gamma]}; l\in I\})\equiv(\hat{\Phi}^*_Q, \{[\hat{U}^*_l(u_0; Q, \Gamma, \varepsilon); u_0, \varepsilon]; l\in I)$ by the norms $\|\cdots\|^*_{pi,k_i}$. $\hat{F}_R[Q]\equiv\{(\hat{\Phi}_Q, \{\hat{\mathfrak{V}}^{(l)}_{[Q,\Gamma]}; l\in I_\Gamma\}); \Gamma\}$ etc. are also defined by using $I_\Gamma\equiv\bigcup_n\{j; 1\leqslant j\leqslant\Gamma(n)\}$.

(ii) Let $\check{\Phi}_Q\equiv\bigcup_{i\in I} B^{\text{loc}}_{pi,k_i}(\Omega; L_i, P_i)$ (as a set). Let $\check{\mathfrak{V}}^{(l)}_{[Q,\Gamma]}\equiv[\check{U}_l(u_0; Q, \Gamma, \varepsilon)\equiv\{u; P_i u\in\Phi^{(l)}_Q, \|\varphi_{\nu,i}P_i(u-u_0)\|_{pi,k_i}\leqslant\varepsilon_{\nu,i}$ for any $i\leqslant l, \Gamma(\nu)\leqslant l\}; u_0(\in B^{(l)}_Q), \varepsilon]$. $\check{F}_R[Q, \Gamma]$ denotes the pair $(\check{\Phi}_Q, \{\check{\mathfrak{V}}^{(l)}_{[Q,\Gamma]}; l\in I\})$.

By the same way we can define $\check{F}^*_R[Q, \Gamma]\equiv(\Phi^*_Q, \{\check{\mathfrak{V}}^{(l)*}_{[Q,\Gamma]}; l\in I\})\equiv(\check{\Phi}^*_R, \{[\check{U}^*_l(u_0; Q, \Gamma, \varepsilon); u_0, \varepsilon]; l\in I\})$ by the norms $\|\cdots\|^*_{pi,k_i}$.

The definition of $\{B^{(l)}_Q\}$ is possible under the norms $\|\cdots\|_{p,k}$ and $\|\cdots\|^*_{p,k}$. The use of $B^{(l)}_Q$ and rational $\varepsilon_{\nu,i}$ sometimes makes the construction of ranked space by countable pre-neighbourhoods possible.

Definition I-2. (i) Let $\hat{\Phi}^w_Q\equiv\{\bigcap_{i\in I, p_i\geqslant 2} B^{\text{loc}}_{pi,k_i}(\Omega; L_i, P_i)\}\cap\{\bigcap_{i\in I, 1\leqslant p_i<2} B^{\text{loc}\,*}_{pi,k_i}(\Omega; L_i, P_i)\}$ and let $\hat{\Phi}^s_Q\equiv\{\bigcap_{i\in I, p_i\geqslant 2} B^{\text{loc}\,*}_{pi,k_i}(\Omega; L_i, P_i)\}\cap\{\bigcap_{i\in I, 1\leqslant p_i<2} B^{\text{loc}}_{pi,k_i}(\Omega; L_i, P_i)$. $\hat{F}^w_Q[Q, \Gamma]\equiv(\hat{\Phi}^w_Q, \{\hat{\mathfrak{V}}^{(l),w}_{[Q,\Gamma]}; l\in I\})$ $(\hat{F}^s_R[Q, \Gamma]\equiv(\hat{\Phi}^s_Q, \{\hat{\mathfrak{V}}^{(l),s}_{[Q,\Gamma]}; l\in I\}))$ can be defined by using the norms $\|\cdots\|^*_{pi,k_i}$ for $1\leqslant p_i<2(p_i\geqslant 2)$ and by using the norms $\|\cdots\|_{pi,k_i}$ for $p_i\geqslant 2(1\leqslant p_i<2)$.

(ii) $\check{F}^w_Q[Q, \Gamma]\equiv(\check{\Phi}^w_Q, \{\check{\mathfrak{V}}^{(l),w}_{[Q,\Gamma]}; l\in I\})$ and $\check{F}^s_R[Q, \Gamma]\equiv(\check{\Phi}^s_Q, \{\check{\mathfrak{V}}^{(l),s}_{[Q,\Gamma]}; l\in I\})$ can be defined by the similar method as (i) using $\bigcup_{i\in I, p_i\geqslant 2}$ etc. instead of $\bigcap_{i\in I, p_i\geqslant 2}$ etc. This $\check{F}^w_R[Q, \Gamma]$ is the widest space.

Theorem I-1. *Each one of* $\hat{F}_R[Q, \Gamma], \check{F}_R[Q, \Gamma], \hat{F}^*_R[Q, \Gamma], \check{F}^*_R[Q, \Gamma], \hat{F}^w_R[Q, \Gamma], \hat{F}^s_R[Q, \Gamma], \check{F}^w_R[Q, \Gamma]$ *and* $\check{F}^s_R[Q, \Gamma]$ *is a ranked space.*

Proof. Since for any $v_0\in B^{(l)}_Q$ and for any positive rational number's sequence $\varepsilon\equiv\{\varepsilon_{\nu,i}\}$ there exist $w_0\in B^{(l+1)}_Q$ and $\varepsilon'\equiv\{\varepsilon'_{\nu,i}\}$ (satisfying $\varepsilon'_{\nu,i}\in(0, \varepsilon_{\nu,i})$) such that $\hat{U}_l(v_0; Q, \Gamma, \varepsilon)\supseteq\hat{U}_{l+1}(w_0; Q, \Gamma, \varepsilon')$ holds from the property of $B^{(l)}_Q$, then $\hat{F}_R[Q, \Gamma]$ becomes a ranked space. By the same way other spaces also become ranked spaces.

Let I be a totally ordered set consisting of limit or isolated ordinal numbers smaller than an (inaccessible) number ω_ν, let $I'\subseteq I$, and let $I''=\bigcup_{j=1}^{\infty}\{i; 1\leqslant i\leqslant i_j, i_j\in I\}\subseteq I'$.

Let $\hat{F}_R[\Omega, \{p_i, k_i\}, \Gamma] \equiv \hat{F}_R[Q, \Gamma]$ by $Q = Q(c) \equiv \{Q_i(c)\} \equiv \{[p_i, k_i, \tilde{C}_0^\infty(\Omega), 1]\}$ and by $\varepsilon_{\nu,i} \equiv \varepsilon$ for ${}^\forall\nu, {}^\forall i$ etc. Let $\mathfrak{B}\{u_i\} = [\{u_i ; i \in I', i \geqslant l\} ; l \in I']$.

Theorem I-2. (i) *If $\{u_i ; i \in I'\}$ tends to u in $\hat{F}_R[\Omega, \{p_i, k_i\}, \Gamma]$ uniquely, $\mathfrak{B}\{u_i\}$ becomes a filter base and tends to u in $\mathfrak{F}(\Omega) \equiv \bigcap_{i \in I} B^{\mathrm{loc}}_{p_i\, k_i}(\Omega)$.*

(ii) *Suppose that $\mathfrak{B}\{u_i\}$ tends to u in $\mathfrak{F}(\Omega)$, and that for any $l \in I''$ and for any $\varepsilon > 0$ there exists a pair $\{\gamma(l), \bar{l}(l, \varepsilon)\}$ of mappings satisfying the following conditions* (**1°**), (**2°**). *Here γ is an one-to-one monotone increasing mapping from I'' to a subset of I, and $\bar{l}(l, \varepsilon)$ is a mapping from $I'' \times \{\varepsilon ; \varepsilon > 0\}$ to I'.* (**1°**) $\sup\{\|\varphi_\nu(u_i - u)\|_{p_\mu, k_\mu} ; \Gamma(\nu), \mu \leqslant \gamma(l)\ (\mu \in I, \nu = 1, 2, \cdots), i \geqslant \bar{l}(l, \varepsilon)\} < \varepsilon$. (**2°**) $\bigcup_{n \in I''} \{i ; 0 < i \leqslant \gamma(n)\} = I$. *If* (**3°**) $\{w ; w \in \Phi^{(l)}_{Q(c)}, \|\varphi_\nu(w - u)\|_{p_\mu, k_\mu} < \varepsilon$ *for* $\Gamma(\nu), \mu \leqslant l\ (\mu \in I, \nu = 1, 2, \cdots)\} \cap B^{(l)}_{Q(c)} \neq \emptyset$ *holds for any $l \in I$ and for any $\varepsilon > 0$, then $u_i (i \in I')$ tends to u uniquely in $\hat{F}_R[\Omega, \{p_i, k_i\}, \Gamma]$.*

Proof. (i) Let $I[\gamma_1]$ be a subset of I and γ_1 be a monotone increasing mapping from $I[\gamma_1]$ satisfying $\bigcup_{l \in I[\gamma_1]} \{i ; 1 \leqslant i \leqslant \gamma_1(l)\} = I$. If $\{u_i ; i \in I'\}$ tends to u in $\hat{F}_R[\Omega, \{p_i, k_i\}, \Gamma]$, there exists a Cauchy sequence $\{\hat{U}_{\gamma_1(l)}(\tilde{u}_l ; Q(c), \Gamma, \{\varepsilon^{(l)}\}) ; l \in I[\gamma_1]\}$ (defined by some γ_1 and satisfying $\hat{U}_{\gamma_1(l)}(u ; Q(c), \Gamma, \{\varepsilon^{(l)}\}) \supseteq \hat{U}_{\gamma_1(l')}(\tilde{u}_{l'} ; Q(c), \Gamma, \{\varepsilon^{(l')}\})$ for $l \leqslant l'$) such that the following (a)$\sim$(d) hold [2] p. 4. (a) $\hat{U}_{\gamma_1(l)}(\tilde{u}_l ; Q(c), \Gamma, \{\varepsilon^{(l)}\}) \in \hat{\mathfrak{B}}^{(\gamma_1(l))}_{[Q, \Gamma]}$. (b) For any $l \in I[\gamma_1]$ there exists $(l \leqslant) \lambda = \lambda(l) \in I[\gamma_1]$ such that $\gamma_1(\lambda) < \gamma_1(\lambda + 1)$ hold. (c) There exists a monotone increasing function $\gamma_2(l)$ (in wide sense) from $I[\gamma_1]$ to I' such that $\hat{U}_{\gamma_1(l)}(\tilde{u}_l ; Q(c), \Gamma, \{\varepsilon^{(l)}\}) \ni u_i$ holds for any $i \geqslant \gamma_2(l)\ (i \in I')$. (d) $\bigcap_{l \in I[\gamma_1]} \hat{U}_{\gamma_1(l)}(\tilde{u}_l ; Q(c), \Gamma, \{\varepsilon^{(l)}\}) \equiv \{u\}$. Then $\varepsilon^{(l)} > 0$ is monotone (in wide sense) decreasing as the function of l, and $\mathfrak{B}\{\varepsilon^{(i)}\} \equiv [\{\varepsilon^{(i)} ; i \in I[\gamma_1], i \geqslant l\} ; l \in I[\gamma_1]]$ tends to 0 in R^1. Since $\hat{U}_{\gamma_1(l)}(u ; Q(c), \Gamma, \{2\varepsilon^{(l)}\}) \equiv \{w ; w \in \hat{\Phi}_{Q(c)}, \|\varphi_\nu(w - u)\|_{p_i, k_i} \leqslant 2\varepsilon^{(l)}$ for any $i \leqslant \gamma_1(l), \Gamma(\nu) \leqslant \gamma_1(l)\} \supseteq \hat{U}_{\gamma_1(l)}(\tilde{u}_l ; Q(c), \Gamma, \{\varepsilon^{(l)}\}) \ni u_i$ for $i \geqslant \gamma_2(l)\ (i \in I')$ holds, then $\mathfrak{B}\{u_i\}$ tends to u in $\mathfrak{F}(\Omega)$. Namely the filter made from $\mathfrak{B}\{u_i\}$ contains all neighbourhoods of u by $\|\varphi_\nu \cdots\|_{p_\mu, k_\mu}$ in $\mathfrak{F}(\Omega)$ for any fixed (μ, ν). Even if $\varepsilon^{(l)} = \varepsilon^{(l')}$ and $\tilde{u}_l = \tilde{u}_{l'}$ hold for $l < l' (l, l' \in I[\gamma_1])$, and if $\gamma_1(l) < \gamma_1(l')$, $\hat{U}_{\gamma_1(l)}(\tilde{u}_l ; Q(c), \Gamma, \{2\varepsilon^{(l)}\}) \supset (\neq) \hat{U}_{\gamma_1(l')}(\tilde{u}_{l'} ; Q(c), \Gamma, \{2\varepsilon^{(l')}\})$ holds. If the description of the ranked space by countable pre-neighbourhoods is possible, $I[\gamma_1]$ may become a countable set.

(ii) Let $\{l_{ij} ; i \in I''\}$ be a sequence of limit or isolated ordinal numbers (satisfying $l_{ij} \leqslant l_{i'j}$ for $i < i'$ in I'') such that $I' \ni l_{ij} \geqslant \mathrm{Max}\,[\bar{l}(i, 1/j), i]$ holds.

Since there exists $\tilde{u}_{ij} \in \{w ; w \in \Phi^{(\gamma(i))}_{Q(c)}, \|\varphi_\nu(w - u)\|_{p_\mu, k_\mu} < 1/j$ for $\Gamma(\nu)$, $\mu \leqslant \gamma(i)\ (\mu \in I', \nu$ natural number$)\} \cap B^{(\gamma(i))}_{Q(c)} (i \in I'')$ from (**3°**), $\hat{U}_{\gamma(i)}(\tilde{u}_{ij} ; Q(c), \Gamma, \{2/j\}) \supseteq \{w ; w \in \hat{\Phi}_{Q(c)}, \|\varphi_\nu(w - u)\|_{p_i, k_i} \leqslant 1/j$ for any $i \leqslant \gamma(i), \Gamma(\nu) \leqslant \gamma(i)\} \supseteq \{u_{i'} ; i' \geqslant l_{ij}, i' \in I'\}$ holds from the condition (**1°**) of Theorem I-2 (ii). Since $\hat{U}_{\gamma(i)}(\tilde{u}_{i,4^j} ; Q(c), \Gamma, \{2/4^j\}) \supseteq \hat{U}_{\gamma(\tilde{i})}(\tilde{u}_{i,4^{j+1}} ; Q(c), \Gamma, \{2/4^{j+1}\})\ (i < \tilde{i})$ $(j = 1, 2, \cdots)$ holds, if $\hat{U}_{\gamma(l)}(\tilde{u}_l ; Q(c), \Gamma, \{\varepsilon^{(l)}\}) \equiv \hat{U}_{\gamma(l)}(\tilde{u}_{i,4^{j+1}} ; Q(c), \Gamma, \{2/4^{j+1}\})$

for $i<l\leqslant\tilde{i}$ (the condition of I'' is derived from here), $\{\hat{U}_{r(l)}(\tilde{u}_l; Q(c), \Gamma, \{\varepsilon^{(l)}\}); l\in I''\}$ is a Cauchy sequence in $\hat{F}_R[\Omega, \{p_i, k_i\}, \Gamma]$ such that $\bigcap_{l\in I''}\hat{U}_{r(l)}(\tilde{u}_l; Q(c), \Gamma, \{\varepsilon^{(l)}\})\equiv\{u\}$ and $\hat{U}_{r(l)}(\tilde{u}_l; Q(c), \Gamma, \{\varepsilon^{(l)}\})\supseteq\{u_i; i>l_{i(4^j),4^j}, i\in I'\}$ for $i(4^{j-1})\leqslant l<i(4^j)$ hold, where $i(4^j)=i$ in $\tilde{u}_{i,4^j}$ and $i(4^{j+1})=\tilde{i}$ in $\tilde{u}_{\tilde{i},4^{j+1}}$. Namely, $u_i(i\in I')$ tends to u uniquely in $\hat{F}_R[\Omega, \{p_i, k_i\}, \Gamma]$.

If I' is countable, (**1**°) (**2**°) and (**3**°) naturally hold. If we use $\{\varepsilon_{\nu,i}\}$ not satisfying $\varepsilon_{\nu,i}\equiv\varepsilon$, it seems that we may use an inaccessible number ω_ν.

The ranked description of the following concrete space gives an interpretation to the use of transcendental number.

Example I-1. Let us treat a compact set Ω^2 in R^2 as Ω. Let $\{k_i(\xi); i\in I\}$ be an ordered set made from $\{k_{(m,n,\delta)}(\xi)\equiv 1+\exp\{(\log_e r)(m\sin^2(2\pi n\theta+\delta)+(1+1/m))\}; m, n$ are positive integer, $\delta\in[0,2\pi)\}$ and $p_i\equiv p\geqslant 1$. Let $\Omega(1)=\{(r\cos\theta, r\sin\theta); r>1, \theta\in[0,2\pi)\}$, let $1_{\Omega(1)}(\xi)$ be the characteristic function of $\Omega(1)$ and let $\tilde{1}_{\Omega(1)}(x)$ be its inverse Fourier transform. If $\|\varphi_\nu\cdot\tilde{1}_{\Omega(1)}(x)*u\|_{p,1+\exp\{(\log_e r)\times(1+1/m)\}}$ (for $\varphi_\nu\in\tilde{C}_0^\infty$) is used as the topology $\tau_{(m,n,\delta)\nu}$, for the space $B^{\rm loc}_{p_i,k_i}(\Omega; L_i, P_i)$ by $\|\varphi_\nu\cdot\tilde{1}_{\Omega(1)}(x)*u\|_{p,k(m,n,\delta)(\xi)}$, and if $\bar{B}^{\rm loc}_\tau$ denotes the closure of $B^{\rm loc}_{p,k}$ by τ, $\bigcap_{(m,n,\delta)}\bigcap_{\nu=1}^\infty\bar{B}^{\rm loc}_{\tau(m,n,\delta),\nu}(\Omega; \tilde{C}_0^\infty, \tilde{1}_{\Omega(1)}(x)*)\equiv\bigcap_{m=1}^\infty\bigcap_{\nu=1}^\infty\bar{B}^{\rm loc}_{\tau(m,0,0),\nu}(\Omega; \tilde{C}_0^\infty, \tilde{1}_{\Omega(1)}(x)*)$ holds. $\bigcap_{(m,n,\delta)}B^{\rm loc}_{p,k(m,n,\delta)}(\Omega; \tilde{C}_0^\infty, \tilde{1}_{\Omega(1)}(x)*)$ can be interpreted as the family of such $\{\tau_{(m,n,\delta),\nu}, \bigcap_{m=1}^\infty\bigcap_{\nu=1}^\infty\bar{B}^{\rm loc}_{\tau(m,0,0),\nu}(\Omega; \tilde{C}_0^\infty, \tilde{1}_{\Omega(1)}(x)*)\}$. This interpretation means that the description by the transcendental factors is the set of the descriptions by the suitable countable factors.

§ 2. The space C^∞ and the space of analytic functions.

Theorem I-3. *If I has countable elements, if $p_i\geqslant 1$ and if $k_i(\xi)=(1+|\xi|)^i$, $\mathfrak{F}(\Omega)\equiv\check{F}_R[\Omega, \{p_i, k_i\}, 1]$ ($\equiv C^\infty(\Omega)$ as a set) holds* (cf. *Theorem* I-2).

Proof. We can prove $\mathfrak{F}(\Omega)\equiv\check{F}_R[\Omega, \{p_i, k_i\}, 1]$ by the similar argument to Theorem I-2. Let $p\geqslant 1$. If $k_i(\xi)=(1+|\xi|)^i$, $(1+|\xi|)^j/k_i(\xi)=(1+|\xi|)^{j-i}\in L_{p'}$ is valid for $j\leqslant i-n-1$, and for any p' satisfying $1/p+1/p'=1$. Then $B^{\rm loc}_{p_i,k_i}(\Omega)\subset C^j(\Omega)$, $(j\leqslant i-n-1, p_i\geqslant 1)$ follows from Hörder's inequality etc. (cf. [1] p. 40, p. 44). Because $\xi^\alpha\hat{u}(\xi)=(\xi^\alpha/(1+|\xi|)^i)((1+|\xi|)^i\hat{u}(\xi))$ is integrable for $|\alpha|\leqslant j$. Namely $\bigcap_{i=1}^\infty B^{\rm loc}_{p_i,k_i}(\Omega)\subseteq C^\infty(\Omega)$ holds. Since $C^\infty(\Omega)\subseteq B^{\rm loc}_{p_i,k_i}(\Omega)$ follows from the Fourier invariance of $(\mathfrak{S})$ (cf. [1] p. 37, p. 44 etc.), $C^\infty(\Omega)\subseteq\bigcap_{i=1}^\infty B^{\rm loc}_{p_i,k_i}(\Omega)$ holds. Because $(\mathfrak{S})\subset L_{p,k}\equiv\{v; V$ measurable, $\|kv\|_p<+\infty\}$ holds in the topological sense. Hence $\check{F}_R[\Omega, \{p_i, k_i\}, 1]\equiv\bigcap_{i=1}^\infty B^{\rm loc}_{p_i,k_i}(\Omega)\equiv C^\infty(\Omega)$ holds as a set.

$E^1(\Omega)\equiv\{1_{\omega(x_0,r)}(x); \omega(x_0,r)\equiv\{x; \|x-x_0\|<r\}\Subset\Omega, r\in(0,1]$ rational, x_0 rational point in $\Omega\}$, where $1_A(x)$ is the characteristic function of A. Let $Q=Q(A,\omega(x_0,r))\equiv\{Q^{(A)}_\alpha[\omega(x_0,r)]\}\equiv\{1, 1, 1_{\omega(x_0,r)}(x), D^\alpha\}$, and $\tilde{\Gamma}(\nu)\equiv 1$ for finite ν.

Theorem I-4. *Let* $|\alpha|\equiv\sum_{i=1}^n\alpha_i$. *The Cauchy sequence*

$\check{U}_1(0; Q(A, \omega(x_0, r)), \tilde{\Gamma}, \{A^{|\alpha|+1}|\alpha|!\}) \supseteq \check{U}_2(0; Q(A, \omega(x_0, r)), \tilde{\Gamma}, \{A^{|\alpha|+1}|\alpha|!\}) \supseteq \cdots$ in $\check{F}_R[\{1, 1, E^1(\Omega), D^\alpha\}, \tilde{\Gamma}]$ (or $\check{U}_1^*(0; Q(A, \omega(x_0, r)), \tilde{\Gamma}, \{A^{|\alpha|+1}|\alpha|!\}) \supseteq \check{U}_2^*(0; \cdots) \supseteq \cdots$ in $\check{F}_R^*[\{1, 1, E^1(\Omega), D^\alpha\}, \tilde{\Gamma}]$) for $1, 2, \cdots \in I_\alpha$ determines a set of analytic functions on a fixed $\omega(x_0, r)$ correspondent to $A>0$. The similar argument holds in $\hat{F}_R[\{1, 1, E^1(\Omega), D^\alpha\}, \tilde{\Gamma}]$ (or in $\hat{F}_R[\{1, 1, E^1(\Omega), D^\alpha\}, \tilde{\Gamma}]$). Here I_α is the totally ordered set constructed from $\{\alpha\}$.

Proof. Since $\sup_{\omega(x_0,r)}|D^\alpha f| = \|1_{\omega(x_0,r)}(x)D^\alpha f\|_{1,1}^* \leqslant \|1_{\omega(x_0,r)}(x)D^\alpha f\|_{1,1} \leqslant A^{|\alpha|+1}|\alpha|!$ holds for any α, $f \in \bigcap_{l=1}^\infty \check{U}_l(0; Q(A, \omega(x_0, r)), \tilde{\Gamma}, \{A^{|\alpha|+1}|\alpha|!\}) \subseteq \bigcap_{l=1}^\infty \check{U}_l^*(0; Q(A, \omega(x_0, r)), \tilde{\Gamma}, \{A^{|\alpha|+1}|\alpha|!\})$ becomes an analytic function in $\omega(x_0, r)$. By the same way $f \in \bigcap_{l=1}^\infty \hat{U}_l(0; Q(A, \omega(x_0, r)), \tilde{\Gamma}, \{A^{|\alpha|+1}|\alpha|!\} \subseteq \bigcap_{l=1}^\infty \hat{U}_l^*(0; Q(A, \omega(x_0, r)), \tilde{\Gamma}, \{A^{|\alpha|+1}|\alpha|!\})$ is also analytic in $\omega(x_0, r)$. The argument in ( ) for $\check{F}_R^*$ etc. are trivial.

Let $\Omega(\varepsilon) \equiv \{x; x \in \Omega, \text{dist}[\Omega^c, x] > \varepsilon\}$.

Theorem I-5. *Let $r<1$ and $M[x] \equiv \text{Max}[x, 0]$.* (i) *$\bigcap_{l=1}^\infty \check{U}_l^*(0; Q(A, \omega(x_0, r)), \tilde{\Gamma}, \{A^{|\alpha|+1}|\alpha|!\}) \subseteq \bigcap_{l=1}^\infty \check{U}_l^*(0; Q(A, \omega(x_0, M[r-j\varepsilon])), \tilde{\Gamma}, \{A^{|\alpha|+1}|\alpha|!(j\varepsilon)^{-|\alpha|}\})$ and $\bigcap_{l=1}^\infty \hat{U}^*(0; Q(A, \omega(x_0, r)), \tilde{\Gamma}, \{A^{|\alpha|+1}|\alpha|!\}) \subseteq \bigcap_{l=1}^\infty \hat{U}_l^*(0; Q(A, \omega(x_0, M[r-j\varepsilon])), \tilde{\Gamma}, \{A^{|\alpha|+1}|\alpha|!(j\varepsilon)^{-|\alpha|}\})$ for $l \in I_\alpha$ hold for any positive integer j.*

(ii) *$\bigcap_{l=1}^\infty \check{U}_l^*(0; Q(A, \omega(x_0, r)), \tilde{\Gamma}, \{A^{|\alpha|+1}|\alpha|!\}) \subseteq \bigcap_{l=1}^\infty \check{U}_l^*(0; Q(A, \omega(x_0, M[r-|\alpha|\varepsilon])), \tilde{\Gamma}, \{A^{|\alpha|+1}\varepsilon^{-|\alpha|}\})$ and $\bigcap_{l=1}^\infty \hat{U}_l^*(0; Q(A, \omega(x_0, r)), \tilde{\Gamma}, \{A^{|\alpha|+1}|\alpha|!\}) \subseteq \bigcap_{l=1}^\infty \hat{U}_l^*(0; Q(A, \omega(x_0, M[r-|\alpha|\varepsilon])), \tilde{\Gamma}, \{A^{|\alpha|+1}\varepsilon^{-|\alpha|}\})$ for $l \in I_\alpha$ hold. Here $1_{\omega(x_0,r)}(x) \in E^1(\Omega)$, $1_{\omega(x_0,M[r-j\varepsilon])}(x) \in E^1(\Omega(r-M[r-j\varepsilon]))$ and $1_{\omega(x_0,M[r-|\alpha|\varepsilon])}(x) \in E^1(\Omega(r-M[r-|\alpha|\varepsilon]))$ holds in $Q(A, \omega(x_0, r)), Q(A, \omega(x_0, M[r-j\varepsilon]))$ and $Q(A, \omega(x_0, M[r-|\alpha|\varepsilon]))$ respectively.*

Proof. Suppose that $\|1_{\omega(x_0,r)}(x)D^\alpha f\|_{1,1}^* \leqslant A^{|\alpha|+1}|\alpha|!$ (for a fixed $A>0$ and for a fixed x_0) holds for any α. Since $\omega(x_0, r-j\varepsilon)$ is empty unless $j\varepsilon<1$, $\|1_{\omega(x_0,M[r-j\varepsilon])}(x)D^\alpha f\|_{1,1}^* \leqslant A^{|\alpha|+1}|\alpha|!(j\varepsilon)^{-|\alpha|}$ for any α. Then (i) holds.

Furthermore $\|1_{\omega(x_0,M[r-|\alpha|\varepsilon])}(x)D^\alpha f\|_{1,1}^* \leqslant A^{|\alpha|+1}|\alpha|! \leqslant A^{|\alpha|+1}|\alpha|^{|\alpha|}$ holds for any α. Since $\omega(x_0, r-|\alpha|\varepsilon)$ is empty unless $|\alpha|\varepsilon<1$, $\|1_{\omega(x_0,M[r-|\alpha|\varepsilon])}(x)D^\alpha f\|_{1,1}^* \leqslant A^{|\alpha|+1}(|\alpha|\varepsilon)^{|\alpha|}\varepsilon^{-|\alpha|} \leqslant A^{|\alpha|+1}\varepsilon^{-|\alpha|}$ holds for any α. Then (ii) holds.

Let $Q(A, \omega(x_0, r-\varepsilon), 2) \equiv \{2, 1, 1_{\omega(x_0,r-\varepsilon)}(x), D^\alpha\}$. $N_\varepsilon(u) \equiv \|1_{\omega(x_0,r-\varepsilon)}(x)u\|_{2,1} = \|1_{\omega(x_0,r-\varepsilon)}(x)u\|_{2,1}^*$ is used in the definition of $\check{U}_l^*(0; Q(A, \omega(x_0, r-\varepsilon), 2), \tilde{\Gamma}, \{\varepsilon_{\nu,i}\}) \equiv \check{U}_l(0; Q(A, \omega(x_0, r-\varepsilon), 2), \tilde{\Gamma}\{\varepsilon_{\nu,i}\})$ etc. in $\check{F}_R^*[\{2, 1, E^1(\Omega), D^\alpha\}, \tilde{\Gamma}] \equiv \check{F}_R[\{2, 1, E^1(\Omega), D^\alpha\}, \tilde{\Gamma}]$.

Theorem I-6. *If u is determined by the Cauchy sequence $\{\check{U}_l(0; Q(A, \omega(x_0, r-c), 2), \tilde{\Gamma}, \{B^{|\alpha|+1}(|\alpha|/c)^{|\alpha|}\}); l=1, 2, \cdots \in I_\alpha\}$ in $\check{F}_R[\{2, 1, E^1(\Omega), D^\alpha\}, \tilde{\Gamma}]$ (or by the Cauchy sequence $\{\hat{U}_l(0; Q(A, \omega(x_0, r-c), 2), \tilde{\Gamma}, \{B^{|\alpha|+1}(|\alpha|/c)^{|\alpha|}); l=1, 2, \cdots$ in $I_\alpha\}$ in $\hat{F}_R[\{2, 1, E^1(\Omega), D^\alpha\}, \tilde{\Gamma}]$), there exists $C_A>0$ such that u is determined by the Cauchy sequence $\{\check{U}_l^*(0; Q(A, \omega(x_0, r-c')), \tilde{\Gamma}, \{C_A^{|\alpha|}|\alpha|!\}); l=1, 2, \cdots \in I_\alpha\}$ in $\check{F}_R^*[\{1, 1, E^1(\Omega), D^\alpha\}, \tilde{\Gamma}]$ (or by the Cauchy sequence $\{\hat{U}_l^*(0; Q(A, \omega(x_0, r-c')), \tilde{\Gamma}, \{C_A^{|\alpha|}|\alpha|!\})\}$ in $\hat{F}_R^*[\{1, 1, E^1(\Omega), D^\alpha\}, \tilde{\Gamma}]$). Here $C'>C>0$.*

Proof. If $\|1_{\omega(x_0,r-c)}(x)D^\alpha u\|_{2,1} \leqslant B^{|\alpha|+1}(|\alpha|/C)^{|\alpha|}$ holds, application of $\|1_{\omega(x_0,r-c')}(x)D^\alpha u\|_{1,1}^* \leqslant \bar{C} \sum_{|\beta|\leqslant n} \|1_{\omega(x_0,r-c')}(x)D^{\alpha+\beta}u\|_{2,1}$ for $u \in C^\infty(\omega(x_0,r-c'))$ (cf. [1] p. 109) gives $\|1_{\omega(x_0,r-c')}(x)D^\alpha u\|_{1,1}^* \leqslant C_M(B/C)^{|\alpha|}(|\alpha|+n)^{|\alpha|+n}$ with a constant $C_M > 0$.
Since $C_M(B/C)^{|\alpha|}(|\alpha|+n)^{|\alpha|+n} \sim C_M(B/C)^{|\alpha|}e^{|\alpha|+n}/\sqrt{2\pi(|\alpha|+n)} \times (|\alpha|+n)!$ $= C_M(Be/C)^{|\alpha|}e^n\ (|\alpha|+n)(|\alpha|+n-1)\cdots(|\alpha|+1)/\sqrt{2\pi(|\alpha|+n)} \times |\alpha|! < C_A^{|\alpha|}$ $\times|\alpha|!$ holds for sufficiently large $|\alpha|$ and for a given $C_A > 0$, this Theorem I-6 holds.

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