148. *Iterated Loop Spaces*

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The aim of this note is to give conditions under which a space or a map can be de-looped k-times up to homotopy. The duals to Theorems 1 and 2 have been obtained by Berstein-Ganea [2]. Our basic lemma (Lemma 1) allows us to overcome the difficulty which arises in dualizing Theorem 3.3 of T. Ganea [4], thereby obtaining a de-looping theorem for a homotopy $\Omega^k S^k$-space (see Theorem 4).

1. A basic lemma. First we set up some notation and conventions. The spaces we consider are supposed to have the based homotopy type of CW-complexes. We denote the loop and suspension functors by Ω and S. Given a map $u: A \to B$, the fibre $\{(a, \gamma) \in A \times B^I; \gamma(0) = *, \gamma(1) = u(a)\}$ and the cofibre $B \cup_u CA$ are denoted by E_u and C_u respectively. The identity maps $\Omega^k X \to \Omega^k X$ and $S^k X \to S^k X$ yield the canonical adjointness maps $\varepsilon_k: S^k \Omega^k X \to X$ and $\eta_k: X \to \Omega^k S^k X$.

Now given a map $f: \Omega X \to Y$, introduce the homotopy commutative diagram

$$
\begin{array}{ccccccccc}
\Omega X & \xrightarrow{f} & Y & & & & & & \\
\downarrow{\scriptstyle \alpha'} & & \| & & & & & & \\
E_i & \longrightarrow & Y & \xrightarrow{i} & C_f & & & & \\
\downarrow{\scriptstyle \beta'} & & \downarrow{\scriptstyle \alpha} & & \| & & & & \\
\Omega X & \longrightarrow & E_{\varepsilon_1 q} & \longrightarrow & C_f & \xrightarrow{\varepsilon_1 q} & X & & \\
\| & & \downarrow{\scriptstyle \beta} & & \downarrow & & \| & & \\
\Omega X & \xrightarrow{-\Omega j} & \Omega C_{\varepsilon_1 q} & \longrightarrow & E_j & \longrightarrow & X & \xrightarrow{j} & C_{\varepsilon_1 q}
\end{array}
$$

in which the vertical maps are constructed as in p. 132 of [6] using the canonical homotopies, i and j are inclusions and $q: C_f \to S\Omega X$ the map pinching Y to a point. Using the Blakers-Massey theorem (see e.g. Theorem 4.3 of [8]) we have

i) $(\beta\alpha)f \simeq \Omega j$,

ii) the construction of $\beta\alpha$ is functorial,

iii) if f is m-connected, $m \geqq 1$, X is 2-connected and Y is $(n-1)$-connected, $n \geqq 1$, then $\beta\alpha$ is $[m + \min(m, n)]$-connected, j $(m+1)$-connected and $C_{\varepsilon_1 q}$ is $\min(n, 2m+1)$-connected.

Iterating the process for j, we get

Lemma 1. *If $f: \Omega^k X \to Y$ is m-connected such that X is $(k+1)$-*

connected and Y is $(n-1)$-connected, $m\geqq n>k-1\geqq 0$, then there exist an $(n+k-1)$-connected space Z and an $(m+n)$-connected map $g\colon Y\to\Omega^k Z$ such that gf is homotopic to a k-fold loop map. The construction of g is functorial. Further, if $h\colon Y\to\Omega^k V$ with V $(k+1)$-connected is a map such that hf can be de-looped k-times, then there exists a $\lambda\colon Z\to V$ with $(\Omega^k\lambda)g\simeq h$.

2. As an immediate consequence of Lemma 1 we obtain the following two theorems which are dual to Theorems 1.4 and 1.6 of [2], so the proofs are omitted.

Theorem 1. *If X is an $(n-1)$-connected space with $\pi_i(X)=0$ for $i\geqq 3n, n\geqq 2$, such that η_k has a homotopy retraction, then X is homotopy equivalent to a k-fold loop space.*

Remark. Taking $k=1$ in Theorem 1, we recover Theorem C of P. J. Hilton [5].

Theorem 2. *Let $\phi\colon\Omega^k A\to\Omega^k B$ be a homotopy $\Omega^k S^k$-map, i.e. $(\Omega^k\varepsilon_k)(\Omega^k S^k\phi)\simeq\phi(\Omega^k\varepsilon_k)$. If A is $(n-1)$-connected, $n>k+1$, and B is a 1-connected space with $\pi_i(B)=0$ for $i\geqq 3n-2k+1$, then ϕ is de-looped k-times.*

The following theorem extends Theorem 5 of [7].

Theorem 3. *Suppose X and Y are n- and q-connected respectively, $k+2\leqq n\leqq q-2$, such that $\pi_i(X)=0$ if $i\geqq 2n+2-k$ and $\pi_j(Y)=0$ if $j\geqq q+n+2-k$. Then $f\colon\Omega^k X\to\Omega^k Y$ is homotopic to k-fold loop map provided that E_f is of the same homotopy type as a k-fold loop space.*

Proof. Denote by $p\colon\Omega^k E\to\Omega^k X$ the fibre of f. Since p is $(q-k)$-connected and since $\Omega^k X$ is $(n-k)$-connected, it follows from Lemma 1 that there is an $(n+q-2k+1)$-connected map $g\colon\Omega^k X\to\Omega^k Z$ such that $gp\simeq\Omega^k j$ for some $j\colon E\to Z$. Moreover, since $fp\simeq 0$ is de-looped k-times, Lemma 1 gives a map $\lambda\colon Z\to Y$ with $(\Omega^k\lambda)g\simeq f$. Killing the homotopy of Z in dimensions $\geqq n+q-k+2$, we get an $(n+q-k+2)$-connected inclusion $h\colon Z\subset W$, hence $h^*\colon[W,Y]\to[Z,Y]$ is onto. This gives rise to a map $\mu\colon W\to Y$ with $\mu h\simeq\lambda$. On the other hand, since $\varepsilon_k\colon S^k\Omega^k X\to X$ is $(2n+2-k)$-connected and since $\pi_i(Z)=0$ for $2n+2-k\leqq i\leqq n+q-k+1$, we see that $\varepsilon_k{}^*\colon[X,W]\to[S^k\Omega^k X,W]$ is onto, which yields a map $\nu\colon X\to W$ with $\nu\varepsilon_k\simeq$ the adjoint of $(\Omega^k h)g$, whence $\Omega^k\nu\simeq(\Omega^k h)g$. Then $f\simeq\Omega^k(\mu\nu)$ as desired.

3. Homotopy $\Omega^k S^k$-spaces. J. Beck [1] has shown that a $\Omega^k S^k$-space can always be de-looped k-times. We shall prove a theorem for a homotopy analogue (cf. Corollary 11.12 of [9]).

Lemma 2. *Let*

$$\begin{array}{ccc} X & \xrightarrow{f} & A \\ {\scriptstyle g}\downarrow & & \downarrow \\ B & \longrightarrow & L \end{array} \qquad\qquad \begin{array}{ccc} \Omega^k X & \xrightarrow{\Omega^k f} & \Omega^k A \\ {\scriptstyle \Omega^k g}\downarrow & & \downarrow \\ \Omega^k B & \longrightarrow & L' \end{array}$$

denote the weak pushout squares (i.e. $L=C_{f,g}$ in the notation of [8]) *and let $\Psi: L' \to \Omega^k L$ be the canonical map. Suppose X is $(k+1)$-connected, $k \geqq 1$. If f is p-connected and g is q-connected, $p \geqq k+1$, $q \geqq k+1$, then Ψ is $(p+q-2k+1)$-connected.*

Corollary 1. *Let $h: U \to V$ be a p-connected map with $(q-1)$-connected U, $q-1 \geqq k+1$, $p \geqq k+1$. Then the canonical map $\psi: C_{\Omega^k h} \to \Omega^k C_h$ is $(p+q-2k+1)$-connected.*

Lemma 3. *Let $f: A \to B$ be a map and let $\eta_A: A \to \Omega^k S^k A$ and $\eta_B: B \to \Omega^k S^k B$ denote the adjointness maps. If f is m-connected, and if A and B are $(n-1)$-connected, $m \geqq n \geqq 1$, then the induced map $C_{\eta_A} \to C_{\eta_B}$ is $[n+\min(2n, m)]$-connected for $k \geqq 2$ and $(n+m)$-connected for $k=1$.*

Proof. Use Theorem 2.1 of Ganea [3], Corollary 1 and the relative Puppe sequences for $\Omega^{i-1}S^{i-1}A \to \Omega^i S^i A \to \Omega^{i+1}S^{i+1}A$ etc.

We say that X is a *homotopy $\Omega^k S^k$-space* (or *homotopy $\Omega^k S^k$-algebra*) if there is a homotopy retraction $r: \Omega^k S^k X \to X$ of η_k such that $r(\Omega^k \varepsilon_k S^k) \simeq r(\Omega^k S^k r)$.

Theorem 4. *Suppose X is an $(n-1)$-connected homotopy $\Omega^k S^k$-space, $n \geqq 2$. If $\pi_i(X)=0$ for $i \geqq 4n+1$, then X has the homotopy type of a k-fold loop space.*

Proof. Introduce the weak pushout squares

$$\begin{array}{ccc} S^k\Omega^k S^k X & \xrightarrow{\varepsilon_k S^k} & S^k X \\ {\scriptstyle S^k r}\downarrow & & \downarrow{\scriptstyle i'} \\ S^k X & \xrightarrow{i} & L_1 \end{array} \qquad \begin{array}{ccc} \Omega^k S^k \Omega^k S^k X & \xrightarrow{\Omega^k \varepsilon_k S^k} & \Omega^k S^k X \\ {\scriptstyle \Omega^k S^k r}\downarrow & & \downarrow{\scriptstyle j'} \\ \Omega^k S^k X & \xrightarrow{j} & L_2. \end{array}$$

Then we have the maps $\Psi: L_2 \to \Omega^k L_1$, $\Phi: L_2 \to X$ such that $\Psi j = \Omega^k i$, $\Phi j = r$. Since r is $2n$-connected and $\varepsilon_k S^k$ is $(2n+k)$-connected, we see from Lemma 2 that Ψ is $(4n+1)$-connected, which implies $\Theta\Psi \simeq \Phi$ for a map $\Theta: \Omega^k L_1 \to X$. Consider the homotopy commutative diagram

$$\begin{array}{ccccccc} \Omega^k S^k X & \xrightarrow{\eta'} & \Omega^k S^k \Omega^k S^k X & \xrightarrow{\Omega^k \varepsilon_k S^k} & \Omega^k S^k X & & \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle \Omega^k S^k r} & & \downarrow{\scriptstyle j'} & & \\ X & \xrightarrow{\eta} & \Omega^k S^k X & \xrightarrow{j} & L_2 & \xrightarrow{\Psi} \Omega^k L_1 \xrightarrow{\Theta} & X, \end{array}$$

where η' and η denote η_k. Then $\Theta\Psi j\eta \simeq 1$ and $(\Omega^k \varepsilon_k S^k)\eta' \simeq 1$. Let $\rho: C_{\eta'} \to C_\eta$ and $\sigma: C_r \to C_{\Omega^k S^k r}$ denote the induced maps. Since C_ρ is homeomorphic to C_σ by virtue of the 3×3 lemma, and since ρ is $3n$-connected by Lemma 3, we see that σ is $3n$-connected. This shows that the induced map $C_r \to C_{j'}$ is $3n$-connected, since $C_{\Omega^k S^k r} \to C_{j'}$ is a homotopy equivalence. It follows from the 5-lemma that $j\eta$ is $3n$-connected, hence Θ is $(3n+1)$-connected. Applying Lemma 1 to Θ, we get a $(4n+1)$-connected map $X \to \Omega^k Y$, from which the theorem follows.

Remark. By duality we may prove that, if X is an $(n-1)$-connected homotopy $S^k\Omega^k$-*CW* complex with $\dim X \leqq 4n-3k-2$, $n \geqq k+1 \geqq 2$, then X has the homotopy type of a k-fold suspension.

References

[1] J. Beck: On H-spaces and infinite loop spaces. Lecture Notes in Math., **99**, 139–153 (1969).

[2] I. Berstein and T. Ganea: Iterated suspensions. Comment. Math. Helv., **45**, 363–371 (1970).

[3] T. Ganea: On the homotopy suspension. Comment. Math. Helv., **43**, 225–234 (1968).

[4] ——: Cogroups and suspensions. Invent. Math., **9**, 185–197 (1970).

[5] P. J. Hilton: Remark on loop spaces. Proc. Amer. Math. Soc., **15**, 596–600 (1964).

[6] Y. Nomura: On mapping sequences. Nagoya Math. J., **17**, 111–145 (1960).

[7] ——: A generalization of suspension theorems. Nagoya Math. J., **19**, 159–167 (1961).

[8] ——: On extensions of triads. Nagoya Math. J., **27**, 249–277 (1966).

[9] J. Stasheff: H-spaces from a homotopy point of view. Lecture Notes in Math., **161** (1970).