145. *Analogue of Fourier's Method for Korteweg-de Vries Equation*

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1. Introduction. In this paper we study the Korteweg-de Vries (KdV) equation

(1) $u_t - 6uu_x + u_{xxx} = 0 \qquad u = u(t) = u(x,t) \ -\infty < x, t < \infty$

for rapidly decreasing initial data. Gardner, Greene, Kruskal and Miura (G.G.K.M.) [2] have associated one dimensional Schrödinger operators $L_{u(t)} = -(d/dx)^2 + u(t)$ to a solution of (1). They have found a simple formula describing the time variation of scattering data of $L_{u(t)}$. This paper is concerned with converse statement which may be viewed as a non-linear analogue of Fourier's method for solving linear partial differential equations of mathematical physics: Given the initial value one determine the scattering data of $L_{u(0)}$. Define scattering data for each t according to the formula of G.G.K.M. Using inverse scattering theory, one can construct potential $u(x,t)$ with prescribed scattering data for each t. Then $u(x,t)$ is a solution of (1).

Throughout the paper subscripts with independent variables denote partial differentiations. Integrations are taken over $(-\infty, \infty)$ unless explicitly indicated.

2. Preparation from scattering theory. Consider one dimensional Schrödinger equation

(2) $$-\phi_{xx} + u(x)\phi = \zeta^2\phi.$$

Under the assumption that $(1+|x|)u(x)$ is integrable, the inverse scattering theory for (2) has been solved by Marchenko for the half line $(0, \infty)$ and then the case of the infinite interval has been treated by Faddeev [1]. We follow [1] in this paper.

For each $\zeta = \xi + i\eta, \eta \geq 0$, there exist unique solutions $f_\pm(x, \zeta)$ which behave like $\exp(\pm i\zeta x)$ as $x \to \pm\infty$. They are called Jost solutions of (2). Jost solutions are analytic in ζ, $\mathrm{Im}\,\zeta > 0$. If $\zeta = \xi$ non-zero real, then f_+ and its complex conjugate f_+^* are independent solutions of (2). One can express f_- as $f_- = a(\xi)f_+^* + b(\xi)f_+$. $a(\xi)$ is limiting value of a function $a(\zeta)$ analytic in ζ, $\mathrm{Im}\,\zeta > 0$. The (right) reflection coefficient $r(\xi) = b(\xi)a(\xi)^{-1}$ is defined for $\xi \neq 0$ and its absolute value is bounded by 1. $a(\zeta)$ has only a finite numbers of zeros. They are all simple and purely imaginary. We denote them by $i\eta_1, \cdots, i\eta_N$. $f_\pm$ are linearly dependent for $\zeta = i\eta_j$ and are square integrable because of the asymptotic property. Put $c_j^{-1}=\int f_+(x, i\eta_j)^2 dx$. The triplet $s=\{r(\xi), \eta_j, c_j\}$ is called the scattering data of the potential u. The coefficients $a(\zeta)$ and $b(\xi)$ in turn can be uniquely reconstructed by the scattering data.

Put $h_\pm(x,\zeta)=\exp(\mp i\zeta x)f_+(x,\zeta)$. Then $h_\pm$ are expressed as

$$h_\pm(x,\zeta)=1\pm\int_0^{\pm\infty} B_\pm(x,y)\exp(\pm 2i\zeta y)dy. \tag{3$\pm$}$$

Coefficients $a(\zeta)$ and $b(\xi)$ have the following integral representations:

$$a(\zeta)=1-(2i\zeta)^{-1}\int u(y)dy-(2i\zeta)^{-1}\int_0^\infty \Pi_2(y)\exp(2i\zeta y)dy$$

$$\Pi_2(y)=\int u(x)B_-(x,-y)dx$$

$$b(\xi)=(2i\xi)^{-1}\int \Pi_1(y)\exp(-2i\xi y)dy$$

$$\Pi_1(y)=u(y)+\int_y^\infty u(x)B_-(x,y-x)dx.$$

Put

$$F(y)=\pi^{-1}\int r(\xi)\exp(2i\xi y)d\xi$$

$$\Omega(y)=\sum_{j=1}^N c_j\exp(-2\eta_j y)+F(y). \tag{4}$$

Then B_+ satisfies the Marchenko equation

$$B_+(x,y)+\int_0^\infty \Omega(x+y+s)B_+(x,s)ds+\Omega(x+y)=0 \tag{5}$$

and the potential u is reconstructed by the formula

$$u(x)=-\frac{\partial}{\partial x}B_+(x,0).$$

If u is in $\mathcal{S}$, Schwartz space on $(-\infty,\infty)$, then $B_\pm(x,y)$ are infinitely differentiable. All of their derivatives are dominated by functions like $\alpha(x+y)$, where $\alpha(x)$ is bounded, decreasing (increasing) and rapidly decreasing as $x\to\infty$ $(x\to-\infty)$ for $B_+(B_-)$. Infinite differentiability in ξ of $h_\pm(x,\xi), h'_\pm(x,\xi), \xi a(\xi)$ and $\xi b(\xi)$ then follows. $\Pi_1(y)$ is in $\mathcal{S}$ and $\Pi_2(y)$ $(y\geq 0)$ is infinitely differentiable with each derivative rapidly decreasing as $y\to\infty$. So $\xi b(\xi)$ and $r(\xi)$ are in $\mathcal{S}$.

Conversely let s satisfy the condition to be a scattering data of a potential $u(x)$. Moreover let $F(y)$, the Fourier transform of $r(\xi)$, be infinitely differentiable and the conditions

$$\int_a^\infty (1+|x|^n)|F^{(m)}(x)|dx<\infty \tag{6}$$

hold for any m, n, a, together with their analogues for the left reflection coefficient. Then $u(x)$ is in $\mathcal{S}$.

3. Solution of the initial value problem. First put $L_u=-D^2+u$ and

$$B_u=-4D^3+3uD+3Du \qquad (D=d/dx).$$

For one parameter family of potential $u=u(t)=u(x,t)$, let $f_{\pm}(x,\zeta;t)$ and $s(t)$ be corresponding Jost functions and scattering data. Suppose u evolves according to the KdV equation written in the form due to Lax: $dL_u/dt=[B_u, L_u]=B_uL_u-L_uB_u$. Then

$$(f_{\pm})_t-B_uf_{\pm}=4(\pm i\zeta)^3f_{\pm} \tag{7$\pm$}$$

hold. These equations imply that $a(\zeta,t)$ is independent of t and so are its zeros. Formulas of G.G.K.M. [2]

$$r(\xi,t)=r(\xi,0)\exp(8i\xi^3t) \qquad c_j(t)=c_j(0)\exp(8\eta_j^3t)$$

follow (see also [3] and [4]).

Conversely let $u(x)$ be in $\mathcal{S}$ and $s=\{r(\xi),\eta_j,c_j\}$ be its scattering data. Put $r(\xi,t)=r(\xi)\exp(8i\xi^3t)$ and $c_j(t)=c_j\exp(8\eta_j^3t)$. Then for each t, $s(t)=\{r(\xi,t),\eta_j,c_j(t)\}$ satisfies the condition to be scattering data of potential $u(x,t)$ belonging to $\mathcal{S}$. To see this, put $a(\zeta,t)=a(\zeta)$ and $b(\xi,t)=b(\xi)\exp(8i\xi^3t)$ where coefficients $a(\zeta)$ and $b(\xi)$ are associated with s. Since $r(\xi,0)$ is in $\mathcal{S}$, so are $r(\xi,t)$ and its Fourier transform

$$F(y,t)=\pi^{-1}\int r(\xi,t)\exp(2i\xi y)d\xi$$

for each t. Define Ω from $s(t)$ by (4). As $a(\xi,t)+b(\xi,t)$ is smooth at $\xi=0$, inverse problem for $(-\infty,\infty)$ is solvable (Lemma 3.1 in [1]).

Next we prove the equation (7+). Put $h(x,\zeta;t)=\exp(-i\zeta t)f_{+}(x,\zeta;t)$ and let $B=B(x,y;t)$ be connected with h by the formula (3+). Then (7+) is equivalent to $h_t=g(x,\zeta;t)$ where

$$g=12\zeta^2h_x-12i\zeta h_{xx}-4h_{xxx}+6u(i\zeta h+h_x)+3u_xh.$$

By direct calculation, we have

$$g=g(x,\zeta;t)=\int_0^\infty C(x,y;t)\exp(2i\zeta y)dy$$

where

$$C=-B_{xxx}+3uB_x.$$

We obtain an integral equation for the kernel C:

$$C(x,y;t)+\int_0^\infty \Omega(x+y+s;t)C(x,s;t)ds$$
$$=\int_0^\infty \Omega_{xxx}(x+y+s;t)B(x,s;t)ds+\Omega_{xxx}(x+y;t).$$

We get the same integral equation for B_t by differentiation of Marchenko equation (5) with respect to t and the identity $\Omega_t+\Omega_{xxx}=0$. Therefore the kernel B_t-C is a solution of homogenous equation associated with (5) known to have only trivial solution. $h_t=g$ and thus (7+) are established. (7$\pm$) are in turn rewritten as $(dL_u/dt-[B_u,L_u])f_{\pm}=0$. Consequently u is a solution of the KdV equation.

Details of proof will be published elsewhere.

Remark. Analogous result has been also formulated in Zakharov and Faddeev [5] by a different method.

References

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