142. On the Automorphism Group of the Full Transformation Semigroups

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Let $\Sigma(\Omega)$ be the semigroup of all the mapping of a set Ω into Ω and $\mathfrak{S}(\Omega)$ the symmetric group on Ω . For $a \in \Omega$ we denote by ε_a the element of $\Sigma(\Omega)$ such that $\varepsilon_a(x) = a$ for every $x \in \Omega$. Let $E = \{\varepsilon_a \mid a \in \Omega\}$.

The purpose of this note is to prove the following:

Theorem. Let S be a subsemigroup of $\Sigma(\Omega)$ containing E and let θ be an isomorphism of S into $\Sigma(\Omega)$. If $\theta(S)$ contains E, then θ can be extended to an inner automorphism of $\Sigma(\Omega)$, i.e., there is an element σ of $\mathfrak{S}(\Omega)$ such that $\theta(\gamma) = \sigma \gamma \sigma^{-1}$ for all $\gamma \in S$. In case Ω is finite, the weaker assumption that $\theta(S) \cap E \neq \phi$ implies the same conclusion.

Denote by $\Sigma^*(\Omega)$ the subsemigroup $\Sigma(\Omega) - \mathfrak{S}(\Omega)$ of $\Sigma(\Omega)$. From the theorem we obtain, immediately the following:

Corollary¹⁾. Every automorphism of $\Sigma(\Omega)$ or of $\Sigma^*(\Omega)$ is inner. Thus the automorphism groups of $\Sigma(\Omega)$ and $\Sigma^*(\Omega)$ are both isomorphic to $\mathfrak{S}(\Omega)$.

Before proving the theorem, we remark that an element ε_a of E satisfies the following equations for all $\gamma \in \Sigma(\Omega)$:

$$(1)$$
 $\varepsilon_a \gamma = \varepsilon_a$

$$\gamma \varepsilon_a = \varepsilon_{r(a)}$$

It follows from (2) that if an element δ of $\Sigma(\Omega)$ satisfies $\delta \varepsilon_a = \delta$ for some $\alpha \in \Omega$, then $\delta = \varepsilon_{\delta(\alpha)}$ is an element of E.

Proof of theorem. Let θ be an isomorphism of S into $\Sigma(\Omega)$ such that $\theta(S) \cap E \neq \phi$. Then there is an element $\beta \in S$ such that $\theta(\beta) = \varepsilon_b$ for some $b \in \Omega$. Since $\varepsilon_a \beta = \varepsilon_a$ for every $a \in \Omega$, $\theta(\varepsilon_a) \varepsilon_b = \theta(\varepsilon_a)$. By the above remark $\theta(\varepsilon_a) \in E$ and hence $\theta(E) \subset E$. If Ω is finite then this implies $\theta(E) = E$. In the case Ω is infinite, if we assume that $\theta(S) \supset E$, then in the same way as above, we get $\theta^{-1}(E) \subset E$ and hence $\theta(E) = E$. Let $\theta(\varepsilon_a) = \varepsilon_{\sigma(a)}$. Then, since $\theta(E) = E$, it follows that σ is in $S(\Omega)$. Now from (2) we have

$$\theta(\gamma)\varepsilon_{\sigma(a)} = \varepsilon_{\sigma\gamma(a)} = \varepsilon_{\theta(\gamma)\sigma(a)}$$

$$= \varepsilon_{\theta(\gamma)\sigma(a)}$$

for any $\gamma \in S$. Thus $\theta(\gamma)\sigma = \sigma \gamma$ and hence $\theta(\gamma) = \sigma \gamma \sigma^{-1}$.

Remark. The following two examples show that the assumptions in the theorem are not superfluous.

¹⁾ This result was first obtained by the first author by making use of the result in [1].

Example 1. Let $\Omega = \{1, 2, 3, 4\}$, and consider the following elements of $\Sigma(\Omega)$;

$$f_1: \{1, 2\} \rightarrow 1,$$
 $\{3, 4\} \rightarrow 3$
 $f_2: \{1, 2\} \rightarrow 1,$ $\{3, 4\} \rightarrow 4$
 $f_3: \{1, 2\} \rightarrow 2,$ $\{3, 4\} \rightarrow 3$
 $f_4: \{1, 2\} \rightarrow 2,$ $\{3, 4\} \rightarrow 4$

Then $f_a f_b = f_a$. Thus the mapping $\theta: E \to \Sigma(\Omega)$ defined by $\theta(\varepsilon_a) = f_a$ is a monomorphism. Now $\theta(E) \cap E = \phi$ and obviously θ can not be extended to an inner automorphism of $\Sigma(\Omega)$.

Example 2. Let $\Omega = \{1, 2, 3, \dots\}$, and let $\theta \colon E \to \Sigma(\Omega)$ be the monomorphism defined by $\theta(\varepsilon_a) = \varepsilon_{a+1}$. Then $\theta(E) \subset E$ but $\theta(E) \neq E$. Thus θ can not be extended to an inner automorphism of $\Sigma(\Omega)$.

Reference

[1] N. Iwahori and N. Iwahori: On a set of generating relations of the full transformation semigroup (to appear).