162. *Countable Structures for Uncountable Infinitary Languages*

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In model theory of infinitary languages with countable conjunctions and finite strings of quantifiers in the sense of H. J. Keisler's book [3], we have some theorems which hold even in the case that there are uncountably many non-logical symbols, e.g. countable isomorphism theorem and countable definability theorem (cf. Scott [4], Chang [1] and Kueker [2]). Of course we have theorems which hold only in the case that there are at most countably many non-logical symbols, e.g. the existence theorem of Scott's sentence (cf. [3]).

In order to make clear the distinction between two kinds of theorems above mentioned we shall show that for each countable structure $\mathfrak{A}$, which is associated to an uncountable infinitary language L, there is a countable sublanguage L_0 of L such that every formula in L is definable in $\mathfrak{A}$ by a formula in L_0. We use the standard model theoretic terminology (cf. [2] and [3]). Let L be a first order language with countable conjunctions and finite strings of quantifiers and possibly uncountably many non-logical symbols. Then we have the following

Theorem. *Let $\mathfrak{A}$ be a countable structure for L. Then there is a countable sublanguage L_0 of L such that for each formula $\varphi(v_1, v_2, \cdots, v_n)$ in L there is a formula $\psi(v_1, v_2, \cdots, v_n)$ in L_0 such that*

$$\mathfrak{A} \models (\forall v_1)(\forall v_2)\cdots(\forall v_n)(\varphi(v_1, v_2, \cdots, v_n) \leftrightarrow \psi(v_1, v_2, \cdots, v_n)).$$

Proof. For each sequence $\sigma = \langle L', a_1, \cdots, a_n \rangle$, where L' a countable sublanguage of L and $a_1, \cdots, a_n$ are elements of $|\mathfrak{A}|$, let φ_σ be the Scott's sentence of the structure $(\mathfrak{A} \restriction L', a_1, \cdots, a_n)$ which is obtained from $\mathfrak{A} \restriction L'$, the reduct of $\mathfrak{A}$ to L', by adding $a_1, \cdots, a_n$ as new individuals. Then there is a formula $\varphi_\sigma(v_1, \cdots, v_n)$ in L' such that $\varphi_\sigma = \varphi_\sigma(a_1, \cdots, a_n)$, i.e. the sentence φ_σ is obtained from the formula $\varphi_\sigma(v_1, \cdots, v_n)$ by replacing $v_1, \cdots, v_n$ by $a_1, \cdots, a_n$ respectively. (We identify the elements a_i in $|\mathfrak{A}|$ and the constant symbols a_i corresponding to them.) Then for each $b_1, \cdots, b_n$ in $|\mathfrak{A}|$, we have

(1) $\mathfrak{A} \models \varphi_\sigma[b_1, \cdots, b_n] \Leftrightarrow (\mathfrak{A} \restriction L', a_1, \cdots, a_n) \cong (\mathfrak{A} \restriction L', b_1, \cdots, b_n).$

Hence if $\sigma_1 = \langle L_1, a_1, \cdots, a_n \rangle$, $\sigma_2 = \langle L_2, a_1, \cdots, a_n \rangle$ and $L_1 \subseteq L_2$, then we have

(2) $\mathfrak{A} \models (\forall v_1)\cdots(\forall v_n)(\varphi_{\sigma_2}(v_1, \cdots, v_n) \rightarrow \varphi_{\sigma_1}(v_1, \cdots, v_n)).$

It follows that for each $a_1, \cdots, a_n$ in $|\mathfrak{A}|$ there is a countable sublanguage L_1 of L such that

(3) $$\mathfrak{A} \models (\forall v_1)\cdots(\forall v_n)(\varphi_{\sigma_1}(v_1, \cdots, v_n) \leftrightarrow \varphi_{\sigma_2}(v_1, \cdots, v_n))$$

holds for every $\sigma_2 = \langle L_2, a_1, \cdots, a_n\rangle$ such that $L_1 \subseteqq L_2$, where $\sigma_1 = \langle L_1, a_1, \cdots, a_n\rangle$, by (2) and the fact that $|\mathfrak{A}|$ is countable. Since there are only countably many finite sequences of elements in $|\mathfrak{A}|$, there is a countable sublanguage L_0 of L such that

(4) $$\mathfrak{A} \models (\forall v_1)\cdots(\forall v_n)(\varphi_{\sigma_1}(v_1, \cdots, v_n) \leftrightarrow \varphi_{\sigma}(v_1, \cdots, v_n)$$

holds for every $a_1, \cdots, a_n$ in $|\mathfrak{A}|$ and for every $\sigma = \langle L_0, a_1, \cdots, a_n\rangle$ and $\sigma_1 = \langle L_1, a_1, \cdots, a_n\rangle$, where $L_0 \subseteqq L_1$. We want to show that this language L_0 satisfies the conclusion of this theorem. Let $\varphi(v_1, \cdots, v_n)$ be a formula in L and $\psi(v_1, \cdots, v_n)$ be the disjunction of all the formulas $\varphi_\sigma(v_1, \cdots, v_n)$ such that σ has the form $\langle L_0, a_1, \cdots, a_n\rangle$ for some $a_1, \cdots, a_n$ such that $\mathfrak{A} \models \varphi[a_1, \cdots, a_n]$. Clearly $\psi(v_1, \cdots, v_n)$ is a formula in L_0. Let L_1 be a countable sublanguage of L such that $L_0 \subseteqq L_1$ and $\varphi(v_1, \cdots, v_n) \in L_1$. Let $b_1, \cdots, b_n$ be elements of $|\mathfrak{A}|$. Then by the definition of $\psi(v_1, \cdots, v_n)$, (1) and (4) we have

$$\begin{aligned}
\mathfrak{A} \models \psi[b_1, \cdots, b_n] &\Leftrightarrow \mathfrak{A} \models \varphi_\sigma[b_1, \cdots, b_n] \text{ for some} \\
&\quad \sigma = \langle L_0, a_1, \cdots, a_n\rangle \text{ such that } \mathfrak{A} \models \varphi[a_1, \cdots, a_n] \\
&\Leftrightarrow \mathfrak{A} \models \varphi_\sigma[b_1, \cdots, b_n] \text{ for some} \\
&\quad \sigma = \langle L_1, a_1, \cdots, a_n\rangle \text{ such that } \mathfrak{A} \models \varphi[a_1, \cdots, a_n] \\
&\Leftrightarrow (\mathfrak{A} \upharpoonright L_1, b_1, \cdots, b_n) \cong (\mathfrak{A} \upharpoonright L_1, a_1, \cdots, a_n) \\
&\qquad \text{for some } a_1, \cdots, a_n \text{ such that } \mathfrak{A} \models \varphi[a_1, \cdots, a_n] \\
&\Leftrightarrow \mathfrak{A} \models \varphi[b_1, \cdots, b_n].
\end{aligned}$$

Hence we have proved

$$\mathfrak{A} \models (\forall v_1)\cdots(\forall v_n)(\varphi(v_1, \cdots, v_n) \leftrightarrow \psi(v_1, \cdots, v_n)). \qquad \text{Q.E.D.}$$

Corollary 1 (*Countable isomorphism theorem, Scott* [4], *Chang* [1]). *Suppose* $\mathfrak{A}_1$ *and* $\mathfrak{A}_2$ *are two countable structures for L. Then we have*

$$\mathfrak{A}_1 \equiv \mathfrak{A}_2 \Leftrightarrow \mathfrak{A}_1 \cong \mathfrak{A}_2.$$

Proof. Let L_1 and L_2 be two countable sublanguages of L for $\mathfrak{A}_1$ and $\mathfrak{A}_2$ stated in our theorem respectively. Let $L_0 = L_1 \cup L_2$ and φ be the Scott's sentence of the structure $\mathfrak{A}_1 \upharpoonright L_0$. Then by our main theorem we have

$$\begin{aligned}
\mathfrak{A}_1 \equiv \mathfrak{A}_2 &\Leftrightarrow \mathfrak{A}_2 \models \varphi \text{ and } \mathfrak{A}_1 \equiv \mathfrak{A}_2 \\
&\Leftrightarrow \mathfrak{A}_1 \upharpoonright L_0 \cong \mathfrak{A}_2 \upharpoonright L_0 \text{ and } \mathfrak{A}_1 \equiv \mathfrak{A}_2 \\
&\Leftrightarrow \mathfrak{A}_1 \cong \mathfrak{A}_2 \qquad \text{Q.E.D.}
\end{aligned}$$

Corollary 2 (*Countable definability theorem, Scott* [4]). *Let* $\mathfrak{A}$ *be a countable structure for L and* $P \subseteqq |\mathfrak{A}|$ *a unary predicate on* $|\mathfrak{A}|$. *Then the following two conditions are equivalent*:

(i) *For any* $Q \subseteqq |\mathfrak{A}|$, $(\mathfrak{A}, P) \cong (\mathfrak{A}, Q)$ *implies* $P = Q$;

(ii) *There is a formula* $\varphi(v)$ *in L such that*

$$(\mathfrak{A}, P) \models (\forall v)(P(v) \leftrightarrow \varphi(v)).$$

Proof. Obviously (ii) implies (i). So it is sufficient to prove that (i) implies (ii). Assume (i) and let L_0 be a countable sublanguage of L for $\mathfrak{A}$ stated in our theorem. Then we have

(iii) $$(\mathfrak{A}, P) \cong (\mathfrak{A}, Q) \Leftrightarrow (\mathfrak{A} \upharpoonright L_0, P) \cong (\mathfrak{A} \upharpoonright L_0, Q)$$

for any $Q \subseteq |\mathfrak{A}|$. Let $\varphi(v)$ be the disjunction of all the formulas $\varphi_a(v)$, where $\varphi_a(a)$ is the Scott's sentence of the structure $(\mathfrak{A} \upharpoonright L_0, a)$ such that $a \in P$. Then we have

$$(\mathfrak{A} \upharpoonright L_0, P) \models (\forall v)(P(v) \leftrightarrow \varphi(v)),$$

by (i) and (iii). Hence we have

$$(\mathfrak{A}, P) \models (\forall v)(P(v) \leftrightarrow \varphi(v)).$$

This means that (i) and (ii) are equivalent. Q.E.D.

Remark *Using our main theorem and Lopez-Escobar's interpolation theorem we can prove Corollary* 2 *above just as we can prove Beth's definability theorem through Craig's interpolation theorem* (*cf. Kueker* [2]).

References

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