161. *On the Boundary Value Problem for Elliptic System of Linear Differential Equations. I*

By Masaki KASHIWARA and Takahiro KAWAI
Research Institute for Mathematical Sciences, Kyoto University
and Département de Mathématiques, Université de Nice

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The purpose of this note is to formulate the boundary value problem for an elliptic system of linear differential equations. We present a theorem which clarifies the relation between the "cohomology groups on the boundary" and those of solutions of the equation on the ambient space (Theorem 1). We refer the reader to Sato, Kawai, and Kashiwara [7] (hereafter referred to as S-K-K) as for notions and notations used in this note. For instance, by a *system* $\mathcal{M}$ of linear differential equations defined on a real analytic manifold M we mean a left $\mathcal{D}_M$-Module which is *admissible*, i.e. admits locally a representation $\mathcal{M}=\mathcal{D}_M \otimes_{\mathcal{D}^f_M} \mathcal{M}^f$ by means of a *coherent* left $\mathcal{D}^f_M$-Module $\mathcal{M}^f$, and by an *elliptic* system a system $\mathcal{M}$ for which $\mathrm{Supp}\,(\mathcal{P}\otimes_{\mathcal{D}}\mathcal{M})\cap\sqrt{-1}S^*M=\phi$. Our formulation of the problem is closely tied with the theory of microfunctions (see Example 1).

Further details of this note will appear elsewhere.

In order to state our main theorem (Theorem 1) we prepare some notations. Let N be a submanifold of a real analytic manifold M with codimension d. Let $\pi_{N/M}: {}^N\widetilde{M}^*\to M$ be the comonoidal transformation of M with center N. Then $\pi^{-1}_{N/M}(N)$ is the conormal spherical bundle $S^*_N M$ by the definition. Let X and Y be a complex neighborhood of M and N respectively. Note that the conormal bundle $T^*_N X$ is a direct sum $T^*_N M\oplus\sqrt{-1}T^*M$ by the complex structure of X. We denote by p the canonical projection from $S^*_N X - S^*_Y X$ to $S^*_N Y=\sqrt{-1}S^*N$ and by q the canonical projection from $S^*_N X-\sqrt{-1}S^*M$ to $S^*_N M$. $\omega_{N/M}$ denotes the locally constant sheaf $\mathcal{H}^d_N(\boldsymbol{C}_M)$.

Theorem 1. *Let $\mathcal{M}$ be an elliptic system of linear differential equations on M. Assume that N is non-characteristic with respect to $\mathcal{M}$. Setting $\mathcal{S}=\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}_M}(\mathcal{M},\mathcal{B}_M)=\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}_M}(\mathcal{M},\mathcal{A}_M)$, we have the following canonical isomorphism:*

$$(1)\qquad \boldsymbol{R}\Gamma_{S^*_N M}(\pi^{-1}_{N/M}\mathcal{S})\otimes\omega_{N/M}[d]\cong\boldsymbol{R}q_*\boldsymbol{R}\,\mathcal{H}om_{p^{-1}\mathcal{P}_N}\Big(\mathcal{P}_{\dot{Y}\to X}\underset{\mathcal{D}_X}{\otimes}\mathcal{M}|_{S^*_N X},\ p^{-1}\mathcal{C}_N\Big)$$

Important applications, which will bring the deep meaning of the theorem tangible, are given in our subsequent papers. It may seem, however, that our formulation above be too general and abstract, so we give in the below three easier examples where Theorem 1 is readily applicable.

Example 1. Let N be a d-dimensional real analytic manifold and M be a complex neighborhood of N. We regard M to be a $2d$-dimensional real analytic manifold and choose as $\mathcal{M}$ the system of Cauchy-Riemann equations on M. Then $\mathcal{S}=\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}_M}(\mathcal{M},\mathcal{B}_M)=\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}_M}(\mathcal{M},\mathcal{A}_M)$ is nothing but the sheaf $\mathcal{O}_M$ of holomorphic functions on M.

Let Z be $\mathrm{Supp}\,(\mathcal{P}_{Y\to X}\otimes_{\mathcal{D}_X}\mathcal{M}|_{S_N^*X})$. Clearly $p: Z\to\sqrt{-1}S^*N$ and $q: Z\to S_N^*M$ are isomorphisms. Moreover $q\circ p^{-1}: \sqrt{-1}S^*N\to S_N^*M\cong\sqrt{-1}S^*N$ turns out to be the identity map. Also we can easily verify that $\mathcal{P}_{Y\to X}\otimes_{\mathcal{D}_X}\mathcal{M}|_{S_N^*X}=p^{-1}\mathcal{P}_{N|Z}$. Hence we have

$$(2)\qquad \boldsymbol{R}q_*\boldsymbol{R}\,\mathcal{H}om_{p^{-1}\mathcal{P}_N}\Big(\mathcal{P}_{Y\to X}\underset{\mathcal{D}_X}{\otimes}\mathcal{M}|_{S_N^*X},\,p^{-1}\mathcal{C}_N\Big)\cong\mathcal{C}_N.$$

Therefore in this case the isomorphism (1) reduces to the definition of $\mathcal{C}_N$, i.e. $\mathcal{C}_N=\boldsymbol{R}\Gamma_{S_N^*M}(\pi_{N/M}^{-1}\mathcal{O}_M)^a\otimes\omega_{N/M}[d]$.

Example 2. Let M be a real analytic manifold and N be a submanifold of M with codimension 1 defined by the equation $\varphi(x)=0$. (We assume that the gradient $\mathrm{grad}\,\varphi(x)$ of $\varphi(x)$ never vanishes on N.) Let $M_\pm$ be the open set $\{x\in M\mid \pm\varphi(x)>0\}$ and $j_\pm$ be the inclusion map from $M_\pm$ into M, respectively. Let $P(x,D_x)$ be a single elliptic differential operator with principal symbol $p_m(x,\zeta)$ and take as $\mathcal{M}$ the system $\mathcal{D}_M/\mathcal{D}_MP$. Then $\mathcal{S}$ is the solution sheaf of $P(x,D_x)u=0$. In this case S_N^*M is decomposed into two parts, namely $N_+=\{(x,\mathrm{grad}_x\,\varphi)\mid x\in N\}$ and $N_-=\{(x,-\mathrm{grad}_x\,\varphi)\mid x\in N\}$. We have $\boldsymbol{R}\Gamma_{S_N^*M}(\pi_{N/M}^{-1}\mathcal{S})^a|_{N_\pm}\cong\boldsymbol{R}\Gamma_{S_N^*M}(\pi_{N/M}^{-1}\mathcal{S})|_{N_\mp}\cong(j_{\pm*}(\mathcal{S})/\mathcal{S})|_N[-1]$.

Let $r=r(x,\eta)$ be the number of roots with negative real part of the equation in τ

$$p_m(x,\tau\,\mathrm{grad}_x\,\varphi(x)+\sqrt{-1}\eta)=0,$$

where $x\in N$ and η is a non-zero real covector not parallel to $\mathrm{grad}_x\,\varphi(x)$. For the sake of simplicity we assume that $r(x,\eta)$ is independent of (x,η). As is well known, this is true if $\dim N>1$ (Then $m=2r$). Under this assumption we easily obtain

$$(3)\qquad \boldsymbol{R}q_*\boldsymbol{R}\,\mathcal{H}om_{p^{-1}\mathcal{P}_N}\Big(\mathcal{P}_{Y\to X}\underset{\mathcal{D}_X}{\otimes}\mathcal{M}|_{S_N^*X},\,p^{-1}\mathcal{C}_N\Big)[-d]=\boldsymbol{R}\pi_{N_*}\mathcal{C}_N^r[-1].$$

One of the fundamental properties of microfunctions asserts that the right hand side of (3) is isomorphic to $(\mathcal{B}_N/\mathcal{A}_N)^r[-1]$. Therefore isomorphism (1) asserts in this case that

$$(4)\qquad (j_{+*}(\mathcal{S})/\mathcal{S})|_N=(\mathcal{B}_N/\mathcal{A}_N)^r$$

holds.

In other words, for any $u\in j_{+*}(\mathcal{S})$ we can define the trace $tr(u)$ of u, which is an r-tuple of microfunctions on N and u can be extended across N if and only if $tr(u)=0$. Thus Theorem 1 generalizes the results on the (elliptic) boundary value problem for a single linear differential equation (in hyperfunction theory). (See Komatsu [3] and Schapira [8] and the references cited there. See also Komatsu, Kawai [4] and Kashiwara [2], where the boundary value problem is treated without the assumption of ellipticity.) The application of Theorem 1 to the so-called "non-elliptic" boundary value problems will be given in our next note.

Example 3. Let M be an n-dimensional complex manifold. A point of M will be denoted by z. As in Example 1 we may regard M as $2n$-dimensional real analytic manifold and take as $\mathcal{M}$ the system of Cauchy-Riemann equations on M. Let N be a real analytic submanifold of codimension 1 defined by the equation $\varphi(z,\bar{z})=0$. (We assume that the gradient $\operatorname{grad}\varphi(z,\bar{z})$ of $\varphi(z,\bar{z})$ never vanishes on N.) $M_{\pm}$ denotes the open subset $\{z\in M \mid \pm\varphi(z,\bar{z})>0\}$ respectively. Denoting $\operatorname{Supp}(\mathcal{P}_{Y\to X}\otimes_{\mathcal{D}_X}\mathcal{M}|_{S_N^*X})$ by Z, we clearly see that $q: Z\to S_N^*M$ is an isomorphism and that $p: Z\to\sqrt{-1}S^*N$ is a closed embedding. Hence we may identify S_N^*M with $p(Z)$. Therefore we have

$$(5) \qquad \boldsymbol{R}q_*\boldsymbol{R}\,\mathcal{H}om_{p^{-1}\mathcal{P}_N}\left(\mathcal{P}_{N\to M}\underset{\mathcal{D}_M}{\otimes}\mathcal{M}, p^{-1}\mathcal{C}_N\right)\cong\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}_N}(\mathcal{M}_N, \mathcal{C}_N).$$

Here the system $\mathcal{M}_N$ signifies $\mathcal{D}_{N\to M}\otimes_{\mathcal{D}_M}\mathcal{M}$ i.e. the system of tangential Cauchy-Riemann equations. We also regard

$$N_{\pm}=\{(x, \pm\operatorname{grad}_x\varphi(x)) \mid x\in N\}$$

as subsets of $\sqrt{-1}S^*N$. Then clearly

$$S_N^*M=N_+\cup N_-=\operatorname{Supp}\left(\mathcal{P}_N\underset{\mathcal{D}_N}{\otimes}\mathcal{M}\right)\cap\sqrt{-1}S^*N.$$

Therefore isomorphism (1) asserts in this case that

$$(6) \qquad \mathcal{H}^k_{\overline{M}_{\pm}}(\mathcal{O}_M)\cong\operatorname{Ext}^{k-1}_{\mathcal{D}_N}(\mathcal{M}_N, \mathcal{C}_N)|_{N_{\pm}}.$$

Note that the generalized Levi form $L(\xi)$ of $\mathcal{M}_N$ on N_+ (cf. S-K-K. Chapter III, Theorem 2.3.10) is

$$\sum_{j,k}\frac{\partial^2\varphi}{\partial z_j\partial\bar{z}_k}\xi_j\bar{\xi}_k \quad \text{with} \quad \sum_j\frac{\partial\varphi}{\partial z_j}\xi_j=0.$$

Therefore the above quoted theorem in S-K-K combined with (6) asserts that

$$(7) \qquad \mathcal{H}^k_{\overline{M}_+}(\mathcal{O}_M)=0$$

if $L(\xi)$ has either at least k-negative eigenvalues or at least $(n+1-k)$ positive eigenvalues. This is the classical theorem proved in Andreotti-Grauert [1]. Maybe Lewy [6] has obtained his celebrated counter-example against solvability of linear differential equation by the aid of such an observation (cf. Lewy [5]).

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