5. On Continuation of Regular Solutions of Partial Differential Equations with Constant Coefficients

By Akira KANEKO*)
University of Tokyo

(Comm. by Kôsaku YOSIDA, M. J. A., Jan. 12, 1973)

This is a short communication of the results of my forthcoming paper [4]. Let $\mathcal{A}$ (resp. $\mathcal{B}$) be the sheaf of real analytic functions (resp. that of hyperfunctions). Let $p(D)$ be a partial differential equation with constant coefficients, and let $\mathcal{A}_p$ (resp. $\mathcal{B}_p$) be the sheaf of real analytic solutions (resp. that of hyperfunction solutions) of $p(D)u=0$. In our earlier work [2], we have given the condition for the operator p in order that $\mathcal{A}_p(U\backslash K)/\mathcal{A}_p(U)=0$, where K is a compact convex subset of $\boldsymbol{R}^n$ and U is one of its open convex neighborhoods. Now let K be the intersection of a compact convex set with the open half space $\{x_n<0\}$ in $\boldsymbol{R}^n$, and let U be one of its open convex neighborhoods. Here, we employ the coordinates $(x_1, \cdots, x_n)=(x', x_n)$ for $\boldsymbol{R}^n$. Concerning the possibility of extension of the solutions of $p(D)u=0$ in $U\backslash K$ to the whole U, we have the following results.

Theorem 1. *$\mathcal{B}_p(U\backslash K)/\mathcal{B}_p(U)=0$ if and only if*

$$H_L(\zeta)\leq\varepsilon|\zeta|+H_{L\backslash K}(\zeta)+C_\varepsilon, \qquad \text{for } \zeta\in N(p),\ ({}^\forall\varepsilon>0,\ {}^\exists C_\varepsilon>0).$$

Here L is the closure of K in $\boldsymbol{R}^n$, $H_L(\zeta)=\sup_{x\in L}\operatorname{Re}\langle x, \sqrt{-1}\zeta\rangle$ is its supporting function and similarly for $H_{L\backslash K}(\zeta)$; $N(p)$ is the characteristic variety $\{\zeta\in C^n;\, p(\zeta)=0\}$ of p.

We can easily prove that the restriction map $\mathcal{B}_p(U)\to\mathcal{B}_p(U\backslash K)$ is injective. Therefore the factor space $\mathcal{B}_p(U\backslash K)/\mathcal{B}_p(U)$ is well defined.

Corollary 2. *If $\mathcal{B}_p(U\backslash K)/\mathcal{B}_p(U)=0$, then p is hyperbolic with respect to the direction $(0, \cdots, 0, 1)$. Conversely, let p be hyperbolic to that direction. Then, for each K which is the part in $\{x_n<0\}$ of a cone with x_n-axis as its axis and with a sufficiently mild vertical angle, we have $\mathcal{B}_p(U\backslash K)/\mathcal{B}_p(U)=0$.*

Here we mean hyperbolicity in the sense of hyperfunctions (see [5], Definition 6.1.1). These results are obtained by cohomological arguements for $\mathcal{B}_p$ and by applying the fundamental principle for $\mathcal{B}_p$ established in [2], II. Note that the possibility of extension of hyperfunction solutions really depends on the shape of K.

As for real analytic solutions we get the following result immediately form Corollary 2, when we take into account the result on

*) Partially supported by Fûjukai.

propagation of regularity $\mathcal{A}_p(U\backslash K)\cap\mathcal{B}_p(U)=\mathcal{A}_p(U)$ ([5], Theorem 5.1.1) and the uniqueness of analytic continuation.

Theorem 3. *Assume that for each irreducible component p_λ of p there exists a sequence of directions $\vartheta_k^{(\lambda)}, k=1,2,\cdots$ converging to $(0,\cdots,0,1)$ such that p_λ is hyperbolic to each direction $\vartheta_k^{(\lambda)}$. Then we have $\mathcal{A}_p(U\backslash K)/\mathcal{A}_p(U)=0$.*

Note that this condition depends on the lower terms of p.

Apart from hyperbolicity, we have the following result.

Theorem 4. *Assume that each irreducible component p_λ of p satisfies either of the following two conditions:*

1) *p_λ satisfies the assumption of Theorem 3.*

2) *There exists a direction $(\vartheta_\lambda, 0)\in \boldsymbol{R}^{n-1}\times\boldsymbol{R}$ with respect to which p_λ is non-characteristic such that $K\subset\{x\in\boldsymbol{R}^n\,;\,\langle\vartheta_\lambda, x'\rangle=0\}$ and any root τ of $p\ (\zeta'+\tau\vartheta_\lambda,\zeta_n)=0$ satisfies the estimate*

$$|\mathrm{Im}\,\tau|\leq\varepsilon|\zeta_n|+b|\mathrm{Im}\,\zeta_n|+C_{\zeta',\varepsilon},\qquad \text{for } \mathrm{Im}\,\zeta_n\geq 0\ (^\forall\varepsilon>0,\ ^\exists C_{\zeta',\varepsilon}>0).$$

Then we have $\mathcal{A}_p(U\backslash K)/\mathcal{A}_p(U)=0$.

The proof of Theorem 4, for the factors corresponding to the condition 2), is carried out employing the represĕntation of $\mathcal{A}_p(U\backslash K)/\mathcal{A}_p(U)$ by a space of holomorphic functions on $N(p)$, and applying a lemma of Phragmén-Lindelöf type. For a compact convex set K, such method is exploited by Grušin [1] and developed in [2]. In our present case, we must employ some type of factor space to represent $\mathcal{A}_p(U\backslash K)/\mathcal{A}_p(U)$. Therefore we need a variant of Phragmén-Lindelöf's theorem in a relative form. In all cases, however, our plan of proof is definite. An intuitive interpretation for it is given in [3].

Note that the condition 2) of Theorem 4 is fairly sharp: we cannot drop the condition of thinness on K in general. In fact, we have the following example: For the wave operator $p(D)=\partial^2/\partial x_1^2-\partial^2/\partial x_2^2-\partial^2/\partial x_n^2$ $(n=3)$ we have the following solution

$$u(x_1,x_2,x_n)=\frac{1}{\sqrt{v(x_1,x_2,x_n)}}\log\left\{\left(\frac{x_2}{v(x_1,x_2,x_n)}\right)^2+\left(\frac{x_n+kx_1-\dfrac{1-k^2}{2k}}{v(x_1,x_2,x_n)}+k\right)^2\right\},$$

where

$$v(x_1,x_2,x_n)=-\left(x_1+\frac{1}{2}\right)^2+x_2^2+\left(x_n+\frac{1}{2k}\right)^2,$$

and $0<k<1$ is a constant. The singularity of u agrees with the following hyperbola near its vertex $(0,0,(1-k^2)/2k)$,

$$\begin{cases} x_2=0\\ x_n^2-x_1^2=\dfrac{1-k^2}{4k^2}.\end{cases}$$

Therefore if K has a positive volume, however small, then we can give a non-trivial element of $\mathcal{A}_p(U\setminus K)/\mathcal{A}_p(U)$ modifying the above solution.

Corollary 5. *Assume that K is contained in the x_n-axis. Then, for any operator $p(D)$ whose principal part does not contain $\partial/\partial x_n$ we have $\mathcal{A}_p(U\setminus K)/\mathcal{A}_p(U)=0$.*

Note that every irreducible component of p enjoys the same property. Thus we can establish the corollary immediately by applying Theorem 4, taking as $(\vartheta_\lambda, 0)$ any non-characteristic direction.

An illustrating example of the operators satisfying the condition of Corollary 5 is the heat equation $\partial/\partial x_n-\Delta_{x'}$ of n-variables. The corresponding result takes a very classical aspect but seems to have been unknown. Note that concerning the heat equation we can apply Theorem 4 directly, taking as $(\vartheta, 0)$ any direction. Thus we have $\mathcal{A}_p(U\setminus K)/\mathcal{A}_p(U)=0$ for any K contained in a perpendicular hyperplane. We further expect that K may be arbitrary in this case.

Errata. Theorem 2.4 and its consequences in my talk at the symposium on "the theory of hyperfunctions and analytic functionals" at RIMS, September 1971, should be replaced by Theorem 4 here and its consequences.

References

[1] Grušin, V. V.: On solutions with isolated singularities for partial differential equations with constant coefficients. Trudy Moskov. Mat. Obšč., **15**, 262–278 (1966).

[2] Kaneko, A.: On continuation of regular solutions of partial differential equations to compact convex sets. J. Fac. Sci. Univ. Tokyo, Sec. 1A, **17**(3), 567–580 (1970); ibid., **18**(3), 415–433 (1972).

[3] ——: Fundamental principle and extension of solutions of partial differential equations with constant coefficients. Proceedings of the symposium on hyperfunctions at Katata, October 1971 (to appear in Springer Lecture Notes in Mathematics).

[4] ——: On continuation of regular solutions of partial differential equations with constant coefficients (submitted to J. Math. Soc. Japan).

[5] Kawai, T.: On the theory of Fourier hyperfunctions and its applications to partial differential equations with constant coefficients. J. Fac. Sci. Univ. Tokyo, Sec. 1A, **17**(3), 467–517 (1970).