1. *A Note on the Selberg Sieve and the Large Sieve*

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1. Let $X \geq 1$ be a real number, and let M, N be integers with $N > 0$. Suppose that there are $\omega(p)$ residue classes $a_{p,j} \pmod p$ $(j=1, \cdots, \omega(p),$ $0 \leq \omega(p) < p)$ corresponding to each prime p, and put $P = \prod_{p \leq X} p$. We define then the integer sequence f_n by

$$f_n = \sum_{p \leq X} P p^{-1} \prod_{j=1}^{\omega(p)} (n - a_{p,j}), \qquad (n = M+1, \cdots, M+N).$$

A. Selberg's sieve method to estimate the quantity

$$Z = \sum_{\substack{n=M+1 \\ (f_n, P)=1}}^{M+N} 1$$

from above is as follows: define multiplicative arithmetic functions ω, Φ, ρ by

$$\omega(d) = \begin{cases} \prod_{p \mid d} \omega(p), & \text{if } d \mid P, \\ 0, & \text{otherwise,} \end{cases}$$

$$\Phi(q) = q \prod_{p \mid q} \left(1 - \frac{\omega(p)}{p}\right),$$

$$\rho(q) = \mu^2(q) \prod_{p \mid q} \frac{\omega(p)}{p - \omega(p)},$$

for natural numbers d, q, and put

$$\lambda_d = \mu(d) \frac{d}{\Phi(d)} \left(\sum_{q \leq X} \rho(q)\right)^{-1} \left(\sum_{\substack{q \leq X/d \\ (q,d)=1}} \rho(q)\right),$$

$$R(d) = \sum_{\substack{n=M+1 \\ d \mid f_n}}^{M+N} 1 - \frac{\omega(d)}{d} N.$$

Then we have

$$Z \leq \sum_{n=M+1}^{M+N} \left(\sum_{d \mid f_n} \lambda_d\right)^2 = \frac{N}{\sum_{q \leq X} \rho(q)} + \sum_{d_1 \leq X} \sum_{d_2 \leq X} \lambda_{d_1} \lambda_{d_2} R([d_1, d_2]), \tag{1.1}$$

where $[d_1, d_2]$ denotes the least common multiple of d_1, d_2, [5].

On the other hand the large sieve method [1], [4] gives

$$\left| \sum_{\substack{n=M+1 \\ (f_n, P)=1}}^{M+N} a_n \right|^2 \sum_{q \leq X} \rho(q) \leq (N + 2X^2) \sum_{\substack{n=M+1 \\ (f_n, P)=1}}^{M+N} |a_n|^2$$

for arbitrary complex numbers a_n $(n = M+1, \cdots, M+N)$, so that we have

$$Z \leq \frac{N + 2X^2}{\sum_{q \leq X} \rho(q)}, \tag{1.2}$$

putting $a_n=1$ for all n.

Although Selberg's method and the large sieve method give quite similar results (1.1), (1.2), the connection between the two methods has not been clear. The purpose of the present paper is to note that the large sieve method can be understood as an algorithm which gives a non-trivial estimate of the sum appearing in (1.1). Namely, we shall prove

$$\left|\sum_{d_1\le X}\sum_{d_2\le X}\lambda_{d_1}\lambda_{d_2}R([d_1,d_2])\right|\le\frac{CX^2}{\sum_{q\le X}\rho(q)} \tag{1.3}$$

with some absolute constant $C>0$. This shows that a remarkable cancellation occurs on the left hand side of this formula, in spite of the way of choice of λ_d which does not care any possibility of cancellations in the sum.

The process proving (1.3) will show at the same time a strong connection between the two sieve methods.

2. Let

$$C_q(n)=\sum_{\substack{a=1\\(a,q)=1}}^{q} e\left(\frac{an}{q}\right)$$

be the Ramanujan sum, and define the multiplicative arithmetic function $C_q^*(n)$ of q by its value

$$C_{p^\alpha}^*(n)=C_{p^\alpha}(n)-\rho(p^\alpha)$$

at a prime power p^α. Throughout this section, we use the conventions

$$a\,\|\,b \qquad \text{for } a\,|\,b,\ \left(a,\frac{b}{a}\right)=1$$

and

$$\sum_{h \bmod q}{}^{*} \qquad \text{for} \sum_{\substack{h \bmod q\\(f(h),q)=1}},$$

where $f(x)=\sum_{p\le x} Pp^{-1}\prod_{j=1}^{\omega(p)}(x-a_{p,j})\in \boldsymbol{Z}[x]$.

Lemma 2.1. *Let k be an integer. Then we have*

$$\sum_{h \bmod p^\alpha}{}^{*} C_{p^\alpha}^*(h-k)=\begin{cases}0, & \text{if } (f(k),p)=1,\\ -p, & \text{if } (f(k),p)\ne 1,\ \alpha=1,\\ 0, & \text{if } (f(k),p)\ne 1,\ \alpha\ge 2.\end{cases}$$

Proof. Suppose that $(f(k),p)=1$. If $\alpha=1$, then

$$\sum_{h \bmod p}{}^{*} C_p^*(h-k)=(p-\omega(p)-1)\left(-1-\frac{\omega(p)}{p-\omega(p)}\right)$$
$$+\left(p-1-\frac{\omega(p)}{p-\omega(p)}\right)=0.$$

If $\alpha\ge 2$, then

$$\sum_{h \bmod p^\alpha}{}^{*} C_{p^\alpha}^*(h-k)=p^\alpha-p^{\alpha-1}\sum_{\substack{h \bmod p^\alpha\\ h\equiv k(p^{\alpha-1})}}{}^{*} 1=0.$$

Next, suppose that $(f(k),p)\ne 1$. If $\alpha=1$, then

$$\sum_{h \bmod p}^{*} C_p^*(h-k)=(p-\omega(p))\left(-1-\frac{\omega(p)}{p-\omega(p)}\right)=-p.$$

If $\alpha\geq 2$, then

$$\sum_{h \bmod p^\alpha}^{*} C_{p^\alpha}^*(h-k)=p^\alpha \sum_{\substack{h \bmod p^\alpha \\ h\equiv k(p^\alpha)}}^{*} 1-p^{\alpha-1} \sum_{\substack{h \bmod p^\alpha \\ h\equiv k(p^{\alpha-1})}}^{*}$$
$$=p^\alpha\cdot 0-p^{\alpha-1}\cdot 0=0.$$

Q.E.D.

Lemma 2.2. *For each positive integer q, it holds that*

$$\rho(q)=\sum_{\substack{a=1 \\ (a,q)=1}}^{q}\left|\Phi(q)^{-1}\sum_{h \bmod q}^{*} e\left(\frac{ah}{q}\right)\right|^2.$$

Proof. The right hand side of the above identity is equal to

$$\Phi(q)^{-2}\sum_{h \bmod q}^{*}\sum_{k \bmod q}^{*} c_q(h-k)$$
$$=\Phi(q)^{-2}\sum_{h \bmod q}^{*}\sum_{k \bmod q}^{*}\sum_{d\|q}\rho(q/d)c_d^*(h-k)$$
$$=\Phi(q)^{-2}\sum_{d\|q}\rho(q/d)\Phi(q/d)^2\sum_{h \bmod d}^{*}\sum_{k \bmod d}^{*} c_d^*(h-k)=\rho(q),$$

by Lemma 2.1.

Q.E.D.

Theorem 2.1. *Let $x_1,\cdots,x_R$ be real numbers with* $\min_{r\neq r'}\|x_r-x_{r'}\|\geq\delta>0$, ($\|z\|=\min_{n\in\boldsymbol{Z}}|z-n|$ for $z\in\boldsymbol{R}$), *and let M,N be integers with $N>0$. Then we have*

$$\sum_{n=M+1}^{M+N}\left|\sum_{r=1}^{R} b_r e(-nx_r)\right|^2=(N+O(\delta^{-1}))\sum_{r=1}^{R}|b_r|^2$$

for any complex numbers b_r, where the implied constant in the O symbol is absolute.

Proof. The inequality

$$\sum_{n=M+1}^{M+N}\left|\sum_{r=1}^{R} b_r e(-nx_r)\right|^2\leq(N+2\delta^{-1})\sum_{r=1}^{R}|b_r|^2$$

follows at once from the formula (2) of [1] using the well known fact that the maximal eigenvalues of A^*A and AA^* coincide, denoting the adjoint of a matrix A by A^*, see also [2]. By means of the same smoothing technique as in [1] we now prove

$$\sum_{n=M+1}^{M+N}\left|\sum_{r=1}^{R} b_r e(-nx_r)\right|^2\geq(N-C\delta^{-1})\sum_{r=1}^{R}|b_r|^2.$$

To show this, we suppose $N\delta$ to be sufficiently large and put $M+1=-[(N-1)/2]=-N_1$, say, without loss of generality. Furthermore we note the following fact: if (c_{ij}) is a positive semi-definite hermitean matrix, then

$$\sum_{i,j} c_{ij}\bar{z}_i z_j\geq\left(\min_i|c_{ii}|-\max_j\sum_{i\neq j}|c_{ij}|\right)\sum_i|z_i|^2$$

for any complex numbers z_i. Using now the function

$$K_m(\alpha)=\sum_{\mu=-m}^{m}(m-|\mu|)e(\mu\alpha)=(\sin\pi m\alpha/\sin\pi\alpha)^2,$$

we can easily see that

$$\sum_{n=M+1}^{M+N}\left|\sum_{r=1}^{R} b_r e(-nx_r)\right|^2 \geq \sum_{r,r'=1}^{R} c_{rr'}\bar{b}_r b_{r'},$$

where

$$c_{rr'}=\frac{1}{L}(K_{N_1}(x_r-x_{r'})-K_{N_1-L}(x_r-x_{r'})), \qquad L=[\delta^{-1}].$$

In fact

$$\begin{aligned}\sum_{n=M+1}^{M+N}\left|\sum_{r=1}^{R} b_r e(-nx_r)\right|^2 &\geq \sum_{|n|\leq N_1-L}\left|\sum_{r=1}^{R} b_r e(-nx_r)\right|^2 \\ &\quad+\sum_{N_1-L<|n|\leq N_1}\frac{N_1-|n|}{L}\left|\sum_{r=1}^{R} b_r e(-nx_r)\right|^2 \\ &=\sum_{r,r'=1}^{R}\bar{b}_r b_{r'}\left(\sum_{|n|\leq N_1}\frac{N_1-|n|}{L}e(n(x_r-x_{r'}))\right. \\ &\quad\left.-\sum_{|n|\leq N_1-L}\frac{N_1-L-|n|}{L}e(n(x_r-x_{r'}))\right).\end{aligned}$$

Therefore, to prove the theorem, it suffices only to show

$$L^{-1}\left(N_1^2-(N_1-L)^2-\max_{r'=1,\cdots,R}\sum_{\substack{r=1\\ r\neq r'}}^{R}|K_{N_1}(x_r-x_{r'})-K_{N_1-L}(x_r-x_{r'})|\right)>N-C\delta^{-1}.$$

But, this follows from

$$\sum_{\substack{r=1\\ r\neq r'}}^{R}|K_{N_1}(x_r-x_{r'})-K_{N_1-L}(x_r-x_{r'})|\leq(2\delta)^{-2}2\sum_{m=1}^{[R/2]}m^{-2}<\frac{\pi^2}{12}\delta^{-2},$$

Since $0\leq K_m(\alpha)\leq(2\|\alpha\|)^{-2}$, $\|x_r-x_{r'}\|\geq\delta$ if $r\neq r'$.

Q.E.D.

We now verify (1.3), which, as mentioned in 1, is the main aim of this paper.

We apply Theorem 2.1 with $\{x_r\}=\{a/q\,;\,1\leq a\leq q\leq X,\ (a,q)=1\}$, so $\delta=X^{-2}$, and put

$$b_{q,a}=\Phi(q)^{-1}\sum_{h \bmod q}^{*} e\left(\frac{ah}{q}\right).$$

Then we obtain

$$\begin{aligned}\sum_{q\leq X}\sum_{\substack{a=1\\(a,q)=1}}^{q} b_{q,a}e\left(-\frac{an}{q}\right)&=\sum_{q\leq X}\Phi(q)^{-1}\sum_{h \bmod q}^{*} c_q(h-n) \\ &=\sum_{q\leq X}\prod_{p^\alpha\|q}\left(\Phi(p^\alpha)^{-1}\sum_{h \bmod p^\alpha}^{*} c_{p^\alpha}(h-n)\right) \\ &=\sum_{q\leq X}\prod_{p^\alpha\|q}\left(\rho(p^\alpha)+\Phi(p^\alpha)^{-1}\sum_{h \bmod p^\alpha}^{*} c_{p^\alpha}^{*}(h-n)\right) \\ &=\sum_{q\leq X}\mu^2(q)\prod_{\substack{p|q\\ p|fn}}\rho(p)\prod_{\substack{p|q\\ p\nmid fn}}(-1)\end{aligned}$$

by Lemma 2.1, and this is equal to

$$\begin{aligned}\sum_{d|fn}\mu(d)\sum_{\substack{q\leq X\\(q,fn)=d}}\mu^2(q)\rho(q/d)&=\sum_{d|fn}\mu(d)\sum_{\substack{m\leq X/d\\(m,fn)=1}}\rho(m) \\ &=\sum_{d|fn}\mu(d)\sum_{\substack{m\leq X/d\\(m,d)=1}}\rho(m)\sum_{\substack{a|fn\\ a|m}}\mu(a) \\ &=\sum_{d|fn}\mu(d)\sum_{a|fn}\sum_{\substack{al\leq X/d\\(al,d)=1\\(a,l)=1}}\mu(a)\rho(a)\rho(l)\end{aligned}$$

$$= \sum_{\substack{kl \leq X \\ k | f_n \\ (k,l)=1}} \rho(l) \sum_{\substack{ad=k \\ (a,d)=1}} \mu(a)\mu(d)\rho(a)$$

$$= \sum_{\substack{kl \leq X \\ k | f_n \\ (k,l)=1}} \rho(l)\mu(k) \prod_{p|k} \frac{p}{p-\omega(p)}$$

$$= \sum_{k|f_n} \mu(k)\frac{k}{\phi(k)} \sum_{\substack{l \leq X/k \\ (l,k)=1}} \rho(l)$$

$$= \Bigl(\sum_{q \leq X} \rho(q)\Bigr) \sum_{d|f_n} \lambda_d.$$

Combining this together with Lemma 2.2, we get

$$\sum_{n=M+1}^{M+N} \Bigl(\sum_{d|f_n} \lambda_d\Bigr)^2 = \frac{N+O(X^2)}{\sum_{q \leq X} \rho(q)}.$$

This and (1.1) prove (1.3).

References*)

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*) As for the proof of Theorem 2.1, see also Montgomery's Springer lecture notes which the author could not read in preparing the present paper.