32. Note on the Results of I. N. Herstein and A. Ramer

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Let A be a division ring which is finite over the center C, and B an intermediate ring of A/C. Let Z be the center of B, and V the centralizer $V_A(B)$ of B in A. In this note, we shall obtain the results of [1] as applications of the following whose proof is obvious by that of [3; Corollary 11.13].

Theorem 1. *Let u be an element of A such that $C[u]$ is a maximal subfield of A. Then, for every non-central element x of A there exists a non-zero y in A such that $A=C[x, yuy^{-1}]=C[y^{-1}xy, u]$.*

In the proof of [3; Corollary 11.13], we used [2; Lemma 1 (i)], which played an essential role in the proof of [1; Theorem 1], too.

Theorem 2. *Let C' be an intermediate ring of Z/C. Then, the following conditions are equivalent:*

(1) *$B=Z$ or $V\neq C'$.*

(2) *C' is a maximal subfield of A or $V\neq C'$.*

(3) *$C'=B\cap M$ for some maximal snbfield M of A.*

Moreover, if one of the above conditions is satisfied then for any maximal subfield $C'[u]$ of A there exists some non-zero y in $V_A(C')$ such that $C'=B\cap C'[yuy^{-1}]$.

Proof. (1)$\Rightarrow$(2): If C' is not a maximal subfield of A and $V=C'$, then $Z=C'\not\subseteq V_A(C')=B$, a contradiction.

(2)$\Rightarrow$(3): It is enough to consider the case $V\neq C'$. We set $A'=V_A(C')$. Then, $C'\subseteq B\subseteq A'$ and C' is the center of A'. Let x be an arbitrary element of $V=V_{A'}(B)$ not contained in C'. As is well-known, A' contains a maximal subfield $C'[u]$ which is a simple extension of C'. Then, by Theorem 1, there exists a non-zero element y in A' such that $A'=C'[x, yuy^{-1}]$. Obviously, $B\cap C'[yuy^{-1}]\subseteq V_{A'}(C'[x])\cap V_{A'}(C'[yuy^{-1}])=V_{A'}(A')=C'$, namely, $B\cap C'[yuy^{-1}]=C'$. It is easy to see that $C'[yuy^{-1}]$ is a maximal subfield of A.

(3)$\Rightarrow$(1): If $B\neq Z$ and $V=C'$, then $C'\subseteq V_A(C')=B$, and hence any maximal subfield M of A containing C' is a maximal subfield of B, which implies $M\cap B=M\neq C'$.

Cororally 1. *The following conditions are equivalent:*

(1) *$B=Z$ or $V\neq Z$.*

(2) *For every intermediate ring C' of Z/C, there exists a maximal subfield M of A such that $C'=B\cap M$.*

If $A \neq B$ then $1 \neq [A: B] = [V: C]$. Hence, by Theorem 2, we have the following

Corollary 2 ([1; Theorem 1]). *Let $A \neq B$. If M is any maximal subfield of A which is a simple extension of C then $C = B \cap yMy^{-1}$ for some non-zero element y in A.*

Now, the results of [1; Corollaries 1, 2, 3] follow immediately from Theorems 1, 2 and Corollary 2 (cf. also [3; Theorem 11.10]). Moreover, as a direct consequence of Theorem 2, we have the following

Corollary 3 ([1; Theorem 2]). *Let C' be a subfield of A containing C. If K is any subfied of A containing C' then $C' = K \cap M$ for some maximal subfield M of A.*

References

[1] I. N. Herstein and A. Ramer: A note on division algebras. Canad. J. Math., **124**, 734–736 (1972).

[2] T. Nagahara: On generating elements of Galois extensions of division rings. Math. J. Okayama Univ., **6**, 181–190 (1957).

[3] H. Tominaga and T. Nagahara: Galois Theory of Simple Rings. Okayama Math. Lectures (1970).