31. *Note on Right-Regular-Ideal-Rings*

By Motoshi HONGAN
Tsuyama Technical College

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Throughout, R is understood to be a ring with 1, which acts as identity on all (right) R-modules. The notation $\cong$ will be used to denote an R-isomorphism between two R-modules. An R-module M is said to be *regular* if there exist some positive integers p, q such that $M^{(p)} \cong R^{(q)}$, where $M^{(p)}$ denotes the direct sum of p copies of M. Following [5], R is called a *right-regular-ideal-ring* (abbr. *right-rir*) if every non-zero right ideal of R is regular. We can define similarly a left-rir, and find a right-rir that is not a left-rir (cf. for instance [4]). As is easily seen, a right-rir is a right Noetherian prime ring, a right Artinian right-rir is simple, and if every non-zero right ideal of R is f.g. (finitely generated) free then R is a right principal ideal domain (cf. [5]).

In what follows, R will represent a right-rir. Let M be a regular R-module. Denoting by $\dim M$ and $\dim R$ the respective dimensions of the R-modules M and R in the sense of Goldie [3; Chapter 3], $M^{(p)} \cong R^{(q)}$ implies $p \cdot \dim M = q \cdot \dim R$, which shows that $r(M) = q/p = \dim M / \dim R$ is an invariant of M. $r(M)$ is called the *rank* of the regular module M. If N is a non-zero submodule of M then, R being right hereditary, N is isomorphic to a finite direct sum of right ideals of R ([1; Theorem I.5.3]). Then, it is easy to see that N is regular. Noting that $\dim M \geqslant \dim N$, we readily obtain $r(M) \geqslant r(N)$. We have proved thus the following which is a sharpening of [5; Corollary to Theorem 2].

Theorem 1. *Let R be a right-rir, and M a regular R-module. If N is a non-zero submodule of M then N is regular and $r(N) \leqslant r(M)$. In particular, $r(\mathfrak{r}) \leqslant 1$ for an arbitrary non-zero right ideal $\mathfrak{r}$ of R.*

Now, it is easy to extend the notion of rank to f.g. R-modules. Let M be an arbitrary f.g. R-module. Then, as is well-known, there exists an exact sequence $0 \to N \to F \to M \to 0$ such that F is f.g. free. (If $N \neq 0$ then N is regular by Theorem 1.) If $0 \to N^* \to F^* \to M \to 0$ is another exact sequence and F^* is f.g. free, then by Schanuel's theorem we have $F \oplus N^* \cong F^* \oplus N$, whence it follows $r(F) - r(N) = r(F^*) - r(N^*)$ ($\geqslant 0$ by Theorem 1), where $r(0) = 0$ by definition. This means that the number $r(M) = r(F) - r(N)$ is independent of the choice of exact sequences. We shall call $r(M)$ the *rank* of M and note that for regular modules this agrees with the rank previously defined. To be easily seen, if M has a finite generating system X of n elements then $n \geqslant r(M) \geqslant 0$ with equality $n = r(M)$ if and only if X is R-free. Finally, we consider an exact sequence $0 \to M' \to M \to M'' \to 0$. Take resolutions of M', M'' using f.g. free modules F', F'', and complete them to a commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & N' & \to & N_0 & \to & N'' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & F' & \to & F_0 & \to & F'' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & M' & \to & M & \to & M'' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

with exact rows and columns, where $F_0 = F' \oplus F''$, and hence $r(F_0) = r(F') + r(F'')$. Since N'' is 0 or regular (Theorem 1), the top row splits, and hence $r(N_0) = r(N') + r(N'')$. We obtain therefore $r(M) = r(F_0) - r(N_0) = r(M') + r(M'')$. This states the following which corresponds to [2; Proposition 2.4]:

Theorem 2. *Given an exact sequence $0 \to M' \to M \to M'' \to 0$ of modules over a right-rir, if M is f.g. then $r(M) = r(M') + r(M'')$.*

References

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