28. Generalized Prime Elements in a Compactly Generated l-Semigroup. I

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In [6] by introducing f-systems authors have defined f-prime ideals in rings as a generalization of prime ideals [2] and s-prime ideals [8], and generalized under certain assumptions usual decomposition theorems of ideals and the concept of relatedness in general rings [2], [3], [7], [8]. The aim of the present note is to present similar results for "elements" of an l-semigroup with some restricted compact generator system. The results obtained here are applicable for general rings and some kind of algebraic systems.

1. Mapping φ, φ-Prime Elements.

Let L be a cm-lattice [1] with the following four conditions:

(1) L has the greatest element e.

(2) L has the least element 0.

(3) Both ae and ea are less than a, i.e. $ae \leq a$ and $ea \leq a$.

(4) L has a compact generator system [4].

It is then easy to see that $a0=0a=0$, $ab \leq a$ and $ab \leq b$ for any a, b in L. If in particular e is unity quantity, the condition (3) is superfluous. From now on Σ will denote a compact generator system of L, $\Sigma(a)$ the set of the compact elements (elements in Σ) which are less than a, and $\Sigma'(a)$ the complement of $\Sigma(a)$ in Σ. Throughout this note we suppose that

(*) *if* $u \in \Sigma(a \cup b)$, *there exists an element* x *of* $\Sigma(a)$ *such that* $\Sigma(x \cup b) \ni u$, *where* a, b *are in* L.

Let R be an associative or nonassociative ring (or more generally a ringoid [1]), and let L_R, Σ_R and Σ_R^* be the sets of all (two-sided) ideals of R, of all principal ideals of R and of all finitely generated ideals of R, respectively. Then it can be shown that L_R is a cm-lattice with (1), (2), (3) and (4). It is easy to see that Σ_R is a compact generator system with the condition (*). Similarly for Σ_R^*. Let G be an arbitrary group, and let L_G, Σ_G and Σ_G^* be the sets of all normal subgroups of G, of all normal subgroups with single generators and of all finitely generated normal subgroups of G, respectively. Then it can be shown that L_G is a cm-lattice under inclusion relation and commutator-product. It is then easily verified that the conditions (1), (2), (3) and (4)

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hold for L_G, and both Σ_G and Σ_G^* are compact generator systems satisfying the condition (∗).

A subset M^* of Σ is called a μ-system [4], iff there is an element z of M^* such that $z \leq xy$ for any two elements x and y in M^*. An element p is prime, iff whenever a product of two elements of L is less than p, then at least one of the factors is less than p. Then it can be proved [4] that p is prime if and only if $\Sigma'(p)$ is a μ-system.

Now we consider a map $\varphi: x \mapsto \varphi(x)$ from Σ into L with the following two conditions:

(1°) *$x \leq \varphi(x)$ for every element $x \in \Sigma$,*

(2°) *$u \leq \varphi(x) \cup a$ implies $\varphi(u) \leq \varphi(x) \cup a$, where $x, u \in \Sigma$ and $a \in L$.*

(1.1) Definition. A subset M of Σ is called a *φ-system,* iff M contains a μ-system M^*, called the *kernel* of M, such that $\Sigma(\varphi(x))$ meets M^* for each element $x \in M$. The void set is a φ-system with void kernel.

(1.2) Definition. An element p of L is said to be *φ-prime,* iff $\Sigma'(p)$ is a φ-system.

For example, the greatest element e is φ-prime.

(1.3) Lemma. *For any φ-prime element $p, \varphi(x_1)\varphi(x_2) \leq p$ implies $x_1 \leq p$ or $x_2 \leq p$.*

Proof. If we suppose that x_i is not less than p for $i=1, 2$, we can take two elements x_1^* and x_2^* in the kernel M^* of $\Sigma'(p)$ such that $x_i^* \leq \varphi(x_i)$ for $i=1, 2$. Choose an element x^* of M^* with $x^* \leq x_1^* x_2^*$. Then we have $x^* \leq \varphi(x_1)\varphi(x_2)$. Hence $\varphi(x_1)\varphi(x_2)$ is not less than p, which is a contradiction.

(1.4) Lemma. *Let M be a φ-system with kernel M^*, and let a be an element of L such that $\Sigma(a)$ does not meet M. Then there exists a maximal element p in the set of the elements b such that $b \geq a$ and $\Sigma(b)$ does not meet M. p is necessarily φ-prime.*

Proof. It is easy to see that the set of the elements b's is inductive. Hence the existence of p follows from Zorn's lemma. To prove that $\Sigma'(p)$ is a φ-system, we consider the set of the elements t of Σ such that $\Sigma(t \cup p)$ meets M^*. Firstly we show the containments $M^* \subseteq T \subseteq \Sigma'(p)$. The containment $M^* \subseteq T$ is trivial. Take any element t of T. Then we can take an element u^* such that $u^* \leq t \cup p$ and $u^* \in M^*$. If we suppose that $t \leq p$, then $u^* \leq p$. This means that M^* meets $\Sigma(p)$, which is a contradiction. Accordingly t is not less than p. Thus we proved the containment $T \subseteq \Sigma'(p)$. Next we will prove that T is a μ-system. Take two arbitrary elements t_1, t_2 of T. Then we can find u_i^* such that $u_i^* \leq t_i \cup p$ and $u_i^* \in M^*$ for $i=1, 2$. Let u^* be an element of M^* with $u^* \leq u_1^* u_2^*$. Then we have $u^* \leq t_1 t_2 \cup p$. By using the condition (∗), we can take an element t such that $u^* \leq t \cup p, t \in \Sigma(t_1 t_2)$. This means that T is a μ-system. Finally we prove that $\Sigma(\varphi(y))$ meets T for each $y \in \Sigma'(p)$. Since $\Sigma(y \cup p)$ meets M, there is an element u of M with $u \leq y \cup p$. Then we have $u \leq \varphi(y) \cup p, \varphi(u) \leq \varphi(y) \cup p$ by (2°). Now we can take an element u^* of M^* such that $u^* \leq \varphi(u)$. Then $u^* \leq \varphi(y) \cup p$. Since there is an element t of $\Sigma(\varphi(y))$ such that $u^* \leq t \cup p$, M^* meets $\Sigma(t \cup p)$, whence t is an element of T. Thus $\Sigma(\varphi(y))$ meets T. Therefore we proved that $\Sigma'(p)$ is a φ-system with kernel T.

(1.5) Definition. A *φ-radical* of an element a of L, denoted by $r_\varphi(a)$, is the supremum (join) of the element x of Σ which have the property that every φ-system containing x meets $\Sigma(a)$.

By using (1.4) we can prove the following theorem, which is similar to the proof of Theorem 1 in [4].

(1.6) Theorem. *The φ-radical of any element a of L is the infimum of all the φ-prime elements containing a.*

Let a be an element of L such that $\Sigma(a)$ does not meet the φ-system M with kernel M^*. Then the family of all φ-systems which contain M^* and does not meet $\Sigma(a)$ is inductive. Hence by Zorn's lemma there exists a maximal μ-system M^* in that family. We now make M_1 as the set of the elements x's of $\Sigma'(a)$ such that $\Sigma(\varphi(x))$ meets M^*. Then evidently M_1 forms a φ-system with kernel M_1^* and does not meet $\Sigma(a)$. By (1.4) there is a φ-prime element p such that $p \geq a$ and $\Sigma(p)$ does not meet M_1. We have proved that $\Sigma'(p)$ is a φ-system with kernel T consisting of the elements t of Σ such that $\Sigma(t \cup p)$ meets M_1^*. Since $T \supseteq M_1^*$, we have $T = M_1^*$. Accordingly, $\Sigma'(p)$ coincides with M_1 by the definition of M_1. In view of this we make the following:

(1.7) Definition. A φ-prime element p is a *quasi-minimal φ-prime element belonging to a*, iff $p \geq a$ and there is a kernel M^* for the φ-system $\Sigma'(p)$ such that M^* is a maximal φ-system which does not meet $\Sigma(a)$.

Let a be any fixed element of L, and let p be a φ-prime element such as $p \geq a$. (The existence of p is assured by *e*.) Then there exists a quasi-minimal φ-prime element p' belonging to a such that $p' \leq p$, which is clear by the above consideration. From (1.6) we obtain the following:

(1.8) Theorem. *The φ-radical of any element in L is represented as the infimum of all quasi-minimal φ-prime elements belonging to a.*

Let A be any two-sided ideal of an associative or nonassociative ring (or ringoid) R. The φ-radical of A and the quasi-minimal φ-prime ideal belonging to A are defined in the obvious way. Similarly for a normal subgroup N of a group G. Then we have the following statements:

The φ-radical of any ideal A of R is represented as the intersection *of all quasi-minimal φ-prime ideals belonging to A.*

The φ-radical of any normal subgroup N of G is represented as the intersection of all quasi-minimal φ-prime normal subgroups belonging to N.

2. φ-Related Elements.

In this section we let L be an associative *cm*-lattice (i.e. *cl*-semigroup [1]) with the conditions (1), (2), (3), (4) and (*). Moreover we assume that *the compact generator system Σ is closed under multiplication.* Then any multiplicatively closed subset of Σ is a φ-system.

If an associative ring (or ringoid) has unity quantity, both Σ_R and Σ_R^* are multiplicatively closed. If G is a group of nilpotent of class 2, L_G is a *cl*-semigroup with the multiplicatively closed system Σ_G.

Following [3], [6], [7] and [8], we define "*φ-related to*" and "*φ-unrelated to*" for elements of L and in particular of Σ.

(2.1) Definition. An element x of Σ is said to be (*left*) *φ-related* to $a \in L$, iff for every x' of $\Sigma(\varphi(x))$ there exists an element u of $\Sigma'(a)$ such that $x'u$ is in $\Sigma(a)$. An element b of L is said to be (*left*) *φ-related* to a, iff every y of $\Sigma(b)$ is φ-related to a. Elements in L (or in Σ) is said to be (*left*) *φ-unrelated* to a, iff they are not φ-related to a.

We can prove easily the following:

(2.2) Lemma. *The set M_φ of all elements which are in Σ and φ-unrelated to a is a φ-system with a multiplicatively closed kernel.*

If the least element 0 is φ-related to each element a of L, then each element of L is φ related to itself, and conversely. For, if we assume that 0 is φ-related to a, then for every $x \in \Sigma(a)$ we have $x \leq a \cup \varphi(0)$, $\varphi(x) \leq a \cup \varphi(0)$. Hence we get $x' \leq a \cup \varphi(0)$ for any x' of $\Sigma(\varphi(x))$. By the condition (*) we can choose two elements $u \in \Sigma(a)$ and $z \in \Sigma(\varphi(0))$ with $x' \leq u \cup z$. Since there is an element v of $\Sigma'(a)$ with $zv \leq a$, we obtain $x'v \leq (u \cup z)v = uv \cup zv \leq av \cup a = a$. Hence x is φ-related to a. Therefore a is φ-related to a. The converse is trivial.

In the rest of this section we assume, as in the case of [5], that *each element of L is φ-related to itself.* Then we obtain

(2.3) Proposition. *The φ-radical $r_\varphi(a)$ of any element a of L is φ-related to a.*

Proof. If there is an element x of $\Sigma(r_\varphi(a))$ which is φ-related to a, then x would be contained in M_φ defined in (2.2). Thus M_φ meets $\Sigma(a)$. This contradicts the above assumption.

Let M_φ be the φ-system defined in (2.2). Then 0 is not contained in M_φ. In other words M_φ does not meet $\Sigma(0) = \{0\}$. Then by (1.4) there exists a maximal element p in the set of all elements $b \in L$ such that $\Sigma(b)$ does not meet M_φ, or equivalently, in the set of all elements φ-related to a. Each such maximal element is necessarily φ-prime.

In view of the above we put the following:

(2.4) Definition. A *maximal φ-prime element belonging to* a is a maximal element in the set of the elements which are φ-related to a.

(2.5) Proposition. *Each element a of L is less than every maximal φ-prime element belonging to a.*

Proof. Let p be any maximal φ-prime element belonging to a. Then it is sufficient to show that $a \cup p$ is φ-related to a. Take an arbitrary element x in $\Sigma(a \cup p)$. Then we have $x \leq u \cup v$ for suitable $u \in \Sigma(a)$ and $v \in \Sigma(p)$. This implies $x \leq u \cup \varphi(v)$, and implies $\varphi(x) \leq u \cup \varphi(v)$. Hence we have $x' \leq u \cup \varphi(v)$ for every x' in $\Sigma(\varphi(x))$. We let v' be an element of $\Sigma(\varphi(v))$ with $x' \leq u \cup v'$. Then since v is φ-related to a, there is an element z of $\Sigma'(a)$ with $v'z \leq a$. We have therefore $x'z \leq (u \cup v')z = uz \cup v'z \leq az \cup a = a$. This means x is φ-related to a. Thus we proved that $a \cup p$ is φ-related to a. q.e.d.

By using (1.4) and considering M_φ in (2.2) we can prove the following:

(2.6) Proposition. *Let a be an element of L. Then every element of Σ or of L which is φ-related to a is less than a maximal φ-prime element belonging to a.*

Let R be an associative ring with unity quantity. An element a of R is called here a left *φ-p divisor of zero* (p for principal), iff the image of the principal (two-sided) ideal (a) by the map φ is (left) φ-related to the zero ideal of R. In particular, if φ is the trivial map $(a) \mapsto (a)$, the left φ-p divisor of zero is the *true left divisor of zero* in the sense of Walt [8]. By using (2.6) we obtain that an element a of R is left φ-p divisor of zero if and only if a is contained in the set-union of the maximal φ-prime ideals belonging to zero.

Principal φ-components of elements in L can be defined naturally. By using (2.6) we obtain decompositions of elements into their principal φ-components, which will be shown in [5].

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