27. On the Euler-Characteristic and the Signature of G-Manifolds*)

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§0. Let W be a closed Riemann surface. A conformal self map of W will be called an automorphism. If G is a finite group of automorphisms of W, then the orbit space W/G is naturally a Riemann surface. In [1], [2] R. D. M. Accola proved certain formulas which relate the genera of W, W/G and W/H where H ranges over certain subgroups of G. He proved them using the Riemann-Hurwitz formula for the coverings $W \to W/G$ and $W \to W/H$.

The purpose of this note is to extend his results. In §1 we shall prove formulas in the case of the Euler-characteristic of compact Hausdorff spaces on which a finite group G acts as the group of homeomorphisms. In §2 we shall prove a formula in the case of the signature of closed connected oriented generalized $4k$-dimensional manifolds over the field of real numbers on which a finite group G acts effectively and orientation preservingly as the group of homeomorphisms.

§1. Throughout this section let X be a compact Hausdorff space on which a finite group G acts and let the cohomology group $H^*(X)$ of X be the Céch cohomology group with real coefficients. Moreover let the groups $H^n(X)$ be finite dimensional, and zero for $n > i$ (i is some integer). Since $H^*(X)$ is naturally a G-module, we have the submodule $H^*(X)^G$ consisting of all invariant elements of $H^*(X)$. Let X/G denote the orbit space and $p: X \to X/G$ the projection. Then the following lemma is known [3].

Lemma 1. *The homomorphism $p^*: H^n(X/G) \to H^n(X)$ is the monomorphism and its image is $H^n(X)^G$.*

Define a homomorphism $\varphi: H^n(X) \to H^n(X)$ by

$$\varphi(\alpha) = \frac{1}{|G|} \sum_{g \in G} g^*(\alpha) \qquad (\alpha \in H^n(X)).$$

Then it is easily seen that α is in $H^n(X)^G$ if and only if $\varphi(\alpha) = \alpha$. Therefore it holds that

$$\begin{aligned} \dim H^n(X)^G &= \operatorname{trace} \varphi \\ &= \frac{1}{|G|} \sum_{g \in G} \operatorname{trace}\,(g^*: H^n(X) \to H^n(X)). \end{aligned}$$

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Denote by $\chi(X)$ the Euler-characteristic of X, and by $\chi(g)$ the Lefschetz number of $g: X\to X$, i.e.

$$\chi(X)=\sum_n (-1)^n \dim H^n(X)$$
$$\chi(g)=\sum_n (-1)^n \operatorname{trace}(g^*: H^n(X)\to H^n(X)).$$

We note that $\chi(X)=\chi(e)$ for the unit element e of G.

Then we have

Lemma 2. *If X is a compact Hausdorff space on which a finite group G acts, then it holds*

$$|G|\,\chi(X/G)=\sum_{g\in G}\chi(g)$$

where $|G|$ is the order of G.

We shall prove

Theorem 1. *Let X be a compact Hausdorff space on which G acts and let $G_1, G_2, \cdots, G_s$ be subgroups of G such that $G=\bigcup_{k=1}^s G_k$. For indices $1\leqq i<j<\cdots<k\leqq s$, put*

$$\chi_{i,j,\dots,k}=\chi(X/(G_i\cap G_j\cap\cdots\cap G_k))$$
$$n_{i,j,\dots,k}=|G_i\cap G_j\cap\cdots\cap G_k|.$$

Put also $\chi_0=\chi(X/G)$, $n_0=|G|$. Then it holds

$$n_0\chi_0=\sum_{1\leqq i\leqq s} n_i\chi_i-\sum_{1\leqq i<j\leqq s} n_{i,j}\chi_{i,j}+\sum_{1\leqq i<j<k\leqq s} n_{i,j,k}\chi_{i,j,k}$$
$$-\cdots-(-1)^s n_{1,2,\dots,s}\chi_{1,2,\dots,s}.$$

Proof. Applying Lemma 2 to G and $G_i\cap G_j\cap\cdots\cap G_k$, we get

$$\chi=n_0\chi_0+q_0$$
$$\chi=n_{i,j,\dots,k}\chi_{i,j,\dots,k}+q_{i,j,\dots,k}$$

where $\chi=\chi(X)$, $q_0=-\sum_{e\neq g\in G}\chi(g)$, $q_{i,j,\dots,k}=-\sum_{e\neq g\in G_i\cap G_j\cap\cdots\cap G_k}\chi(g)$. On the other hand, inductive arguments show

$$q_0=\sum_{1\leqq i\leqq s} q_i-\sum_{1\leqq i<j\leqq s} q_{i,j}+\sum_{1\leqq i<j<k\leqq s} q_{i,j,k}-\cdots-(-1)^s q_{1,2,\dots,s}.$$

Therefore we have

$$\chi-n_0\chi_0=\sum_{1\leqq i<i\leqq s}(\chi-n_i\chi_i)-\sum_{1\leqq i<j\leqq s}(\chi-n_{i,j}\chi_{i,j})$$
$$+\sum_{1\leqq i<j<k\leqq s}(\chi-n_{i,j,k}\chi_{i,j,k})-\cdots-(-1)^s(\chi-n_{1,2,\dots,s}\chi_{1,2,\dots,s}).$$

Since $\chi=\sum_{1\leqq i\leqq s}\chi-\sum_{1\leqq i<j\leqq s}\chi+\sum_{1\leqq i<j<k\leqq s}\chi-\cdots-(-1)^s\chi$, we have the desired results.

A finite group G is said to admit a partition if there is a set $\{G_1, G_2, \cdots, G_s\}$ of subgroups of G ($s\geqq 2$) such that

$$G=\bigcup_{i=1}^s G_i,\qquad G_i\cap G_j=\langle e\rangle\ (i\neq j).$$

Corollary 1. *Let X be a compact Hausdorff space on which a finite group G acts, and assume that G admits a partition $\{G_1, G_2, \cdots, G_s\}$. Then we have*

$$(s-1)\chi(X)+|G|\,\chi(X/G)=\sum_{i=1}^s |G_i|\,\chi(X/G).$$

Proof. Since $G_i \cap G_j = \langle e \rangle$ $(i \neq j)$, if $\#\{i, j, \cdots, k\} \geqq 2$, we have $\chi_{i,j,\dots,k} = \chi(X/G)$, $n_{i,j,\dots,k} = 1$, with the notation of Theorem 1. Therefore the theorem implies

$$|G|\chi(X/G) = \sum_{i=1}^{s} |G_i|\chi(X/G_i) - \left(\sum_{i=2}^{s}(-1)^i \binom{s}{i}\right)\chi(X)$$

or $|G|\chi(X/G) = \sum_{i=1}^{s} |G_i|\chi(X/G_i) + (1-s)\chi(X)$. This completes the proof.

As applications of Corollary 1 we shall consider the dihedral group and the affine transformation group on a finite field. Let D_n be the dihedral group. D_n is a group admitting a partition. In fact, let $R \in D_n$ generate the cyclic subgroup $\langle R \rangle$ of order n and let v be an element of order two not in $\langle R \rangle$. Then $\{\langle R \rangle, \langle v \rangle, \langle Rv \rangle, \cdots, \langle R^{n-1}v \rangle\}$ is a partition of D_n.

Application 1. *Let X be a compact Hausdorff space on which D_n acts. Then we have*

$$\chi(X) + 2\chi(X/D_n) = \chi(X/\langle R \rangle) + \chi(X/\langle v \rangle) + \chi(X/\langle Rv \rangle).$$

Proof. By Corollary 1 we have

$$n\chi(X) + 2n\chi(X/D_n) = n\chi(X/\langle R \rangle) + 2\sum_{i=0}^{n-1} \chi(X/\langle R^i v \rangle).$$

If n is odd, then all subgroups $\langle R^i v \rangle$ are conjugate, and hence

$$\chi(X/\langle R^i v \rangle) = \chi(X/\langle v \rangle) \qquad \text{for } i = 1, 2, \cdots, n-1.$$

If n is even, then $\langle R^i v \rangle$ and $\langle R^j v \rangle$ are conjugate if and only if $i \equiv j$ mod 2. Thus we have

$$\sum_{i=0}^{n-1} \chi(X/\langle R^i v \rangle) = \begin{cases} n\chi(X/\langle v \rangle) & \text{for odd } n \\ \dfrac{n}{2}(\chi(X/\langle v \rangle) + \chi(X/\langle Rv \rangle)) & \text{for even } n. \end{cases}$$

Therefore we have the desired result.

Let $F(q)$ be a finite field of characteristic q. Denote by $f_{(a,b)}$ the correspondence $F(q) \ni x \to ax + b \in F(q)$ $(a, b \in F(q),\ a \neq 0)$. Put $K = \{f_{(a,b)} \mid a,\ b \in F(q),\ a \neq 0\}$, $N = \{f_{(1,b)} \mid b \in F(q)\}$ and $K_0 = \{f_{(a,0)} \mid 0 \neq a \in F(q)\}$. Note that the affine transformation group K on $F(q)$ is a group admitting a partition.

Then we have the following application by the method similar to the proof of Application 1.

Application 2. *Let X be a compact Hausdorff space on which K acts. Then we have*

$$\chi(X) + (q-1)\chi(X/K) = \chi(X/N) + (q-1)\chi(X/K_0).$$

Remark 1. If (X, A) is a topological G-space pair where X is a compact Hausdorff space, A is a closed subspace of X and G is a finite group, we have the isomorphism $p^*: H^n(X/G, A/G) \to H^n(X, A)^G$, where $p: (X, A) \to (X/G, A/G)$ is the projection, by Lemma 1. Thus all results in this section hold also in relative case.

§2. Let M be a $4k$-dimensional closed oriented connected generalized manifold over R [3],[7], and let G be a finite group acting on M effectively by orientation preserving homeomorphisms. The cup product defines a non-degenerate symmetric bilinear form on $H^{2k}(M;R)$. Let H^+ (resp. H^-) be the maximal subspace on which this form is positive (resp. negative) definite. Then we have $H^{2k}(M;R)=H^+\oplus H^-$ and $g^*(H^+)=H^+$, $g^*(H^-)=H^-$ $(g\in G)$. The signature of M is defined to be $\sigma(M)=\dim H^+-\dim H^-$. And for any element $g\in G$, the Atiyah-Singer signature $\sigma(g)$ is defined by $\sigma(g)=\mathrm{trace}\,(g^*|H^+)-\mathrm{trace}\,(g^*|H^-)$. We note that $\sigma(M)=\sigma(e)$ for the unit element e of G.

It is known that M/G is a $4k$-dimensional orientable generalized manifold over R [7]. Therefore we can define as usual the signature $\sigma(M/G)$ of M/G, where we fix the orientation of M/G so that the projection $p: M\to M/G$ is the orientation preserving map.

Then by the method similar to the proof of Lemma 2, we get

$$|G|\,\sigma(M/G)=\sum_{g\in G}\sigma(g) \qquad \text{(see [4]).}$$

Thus we have the following theorem concerning to the signature. The proof is similar to the proof of Theorem 1.

Theorem 2. *Let M be a $4k$-dimensional closed oriented connected generalized manifold over R, and let G be a finite group acting on M effectively by orientation preserving homeomorphisms. Let $G_1, G_2, \cdots, G_s$ be subgroups of G such that $G=\bigcup_{k=1}^{s} G_k$. For indices $1\leqq i<j<\cdots<k\leqq s$, put $\sigma_{i,j,\dots,k}=\sigma(M/(G_i\cap G_j\cap\cdots\cap G_k)$, $n_{i,j,\dots,k}=|G_i\cap G_j\cap\cdots\cap G_k|$. Put also $\sigma_0=\sigma(M/G)$, $n_0=|G|$. Then it holds*

$$n_0\sigma_0=\sum_{1\leqq i\leqq s} n_i\sigma_i-\sum_{1\leqq i<j\leqq s} n_{i,j}\sigma_{i,j}+\sum_{1\leqq i<j<k\leqq s} n_{i,j,k}\sigma_{i,j,k} -\cdots-(-1)^s n_{1,2,\dots,s}\sigma_{1,2,\dots,s}.$$

Remark 2. We remark that the results similar to Corollary 1 and Applications 1, 2 hold in the case of the signature.

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