41. *A Remark on a Sufficient Condition for Hypoellipticity*

By Akira TSUTSUMI
College of General Education, Osaka University

(Comm. by Kinjirô KUNUGI, M. J. A., March 12, 1973)

1. Introduction. Let $P=P(x,D_x)=\sum_{|\alpha|\leq m} a_\alpha(x)D_x^\alpha$ be a differential operator where $x=(x_1,\cdots,x_n)$ is a point of a open subset Ω in real n-space R_x^n, $\alpha=(\alpha_1,\cdots,\alpha_n)$ is a multi-index with its length $|\alpha|=\alpha_1+\cdots+\alpha_n$ and $D_x^\alpha=(-i\partial/\partial x_1)^{\alpha_1}\cdots(-i\partial/\partial x_n)^{\alpha_n}$. For $\xi\in R^n$ we denote $\xi^\alpha=\xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n}$, $|\xi|=(\xi_1^2+\cdots+\xi_n^2)^{1/2}$, $\langle\xi\rangle=1+|\xi|$, $P(x,\xi)=\sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha$ and $P_{(\beta)}^{(\alpha)}(x,\xi)=D_\xi^\alpha(iD_x)^\beta P(x,\xi)$.

Simple and weak sufficient conditions for hypoellipticity are given by L. Hörmander which include not only differential operators but also pseudo-differential operators ([2] § 4 Theorem 4.2, p. 164). In this note we shall give a slightly different sufficient condition for hypoellipticity which is stated by using a basic weight function depending also on the x-variable instead of $\langle\xi\rangle$ only. The usage such a basic weight function is effective for study of asymptotic behavior of spectral function of hypoelliptic differential operator which will appear in a forthcoming paper.

We confine ourselves in case of differential operators but it seems quite possible to extend it in case of pseudo-differential operators, because the proof of the main theorem depends on a construction of a parametrix just along the arguments in [1] and [2]. I wish to thank Mr. M. Nagase for his advice through discussion.

2. Theorem and outline of the proof. Theorem. *Let $P(x,\xi)$ be written in the sum $P(x,\xi)=p_0(x,\xi)+p_1(x,\xi)$ where $p_0=p_0(x,\xi)$ and $p_1=p_1(x,\xi)$ satisfy the following conditions:*

(2.1) *The coefficients are in C^∞.*

For any $x\in\Omega$ and α and β there exist the constants $C_{x,\alpha,\beta}>0$, $C_x>0$, and $A_x>0$ such that

(2.2) $$|p_{0(\beta)}^{(\alpha)}(x,\xi)|\leqq C_{x,\alpha,\beta}|p_0(x,\xi)|^{1-\rho|\alpha|+\delta|\beta|}$$

(2.2)′ $$|p_{1(\beta)}^{(\alpha)}(x,\xi)|\leqq C_{x,\alpha,\beta}|p_0(x,\xi)|^{1-\rho(|\alpha|+1)+\delta(|\beta|+1)} \qquad \text{for } |\xi|\geqq A_x,$$

where ρ and δ are some constants depending only on $P(x,D)$ and satisfying $0\leqq\delta<\rho\leqq 1$,

(2.3) $$|p_0(x,\xi)|\geqq C_x|\xi|^{m'} \quad 0<m'\leqq m, \qquad \text{for } |\xi|\geqq A_x,$$

(2.4) $$m'\delta<1,$$

and $C_{x,\alpha,\beta}$, C_x and A_x are bounded when x is in compact subset of Ω. Then the operator $P(x,D_x)$ is hypoelliptic: $u\in\mathcal{D}'(\Omega)$ satisfying the equa-

tion $P(x, D_x)u=f$ *is in* C^∞ *in any open subset of* Ω *where* f *is in* C^∞. The proof of the theorem is obtained from the following series of lemmas. Let $q_k=q_k(x, \xi)$ $k=0, 1, \cdots$ be defined inductively

(2.5) $$p_0 \cdot q_0 = 1$$

(2.6) $$p_0 \cdot q_k = -p_1 \cdot q_{k-1} - \sum_{\substack{|\alpha|+l=k \\ l<k}} 1/\alpha!\, P^{(\alpha)} \cdot q_{l(\alpha)} \qquad \text{for } |\xi| \geqslant A_x.$$

Lemma 1. *The* q_k, $k=1, 2, \cdots$ *have the following form:*

$$q_{k}{}^{(\alpha)}_{(\beta)} = 1/p_0 \sum_2^{2k+|\alpha|+|\beta|} \prod_{\lambda=1}^{\lambda=i} (P^{(\alpha_\lambda)}/p_0) \prod_{\mu=1}^{\mu=j} (P^{(\alpha'_\mu)}_{(\beta_\mu)}/p_0)$$
$$\cdot \prod_{\nu=1}^{\nu=\kappa} (p_0{}^{(\alpha''_\nu)}_{(\beta'_\nu)}/p_0) \prod_{\iota=1}^{\iota=\tau} (p_1{}^{(\alpha'''_\iota)}_{(\beta''_\iota)}/p_0)$$

for $|\xi| \geqq A_x$, where $\alpha, \beta, \alpha_\lambda, \beta_\mu, \cdots, \alpha'''_\iota$ and β''_ι are multi-indices satisfying $\alpha_\lambda \neq 0$, $\lambda=1, 2, \cdots, i$, $\alpha'_\mu \neq 0$, $\mu=1, 2, \cdots, j$, $|\alpha''_\nu| \geq 0$, $\nu=1, 2, \cdots, \kappa$, $|\alpha'''_\iota| \geq 0$, $\iota=1, 2, \cdots, \tau$, $\beta_\mu \neq 0$, $\mu=1, 2, \cdots, j$, $\beta'_\nu \neq 0$, $\nu=1, 2, \cdots, \kappa$, and $|\beta''_\iota| \geqq 0$, $\iota=1, =1, 2, \cdots, \tau$, and furthermore

$$\Big|\sum_{\lambda=1}^{\lambda=i} \alpha_\lambda + \sum_{\mu=1}^{\mu=j} \alpha'_\mu + \sum_{\nu=1}^{\nu=\kappa} \alpha''_\nu + \sum_{\iota=1}^{\iota=\tau} \alpha'''_\iota\Big| + \tau = k + |\alpha|,$$
$$\Big|\sum_{\mu=1}^{\mu=j} \beta_\mu + \sum_{\nu=1}^{\nu=\kappa} \beta'_\nu + \sum_{\iota=1}^{\iota=\tau} \beta''_i\Big| + \tau = k + |\beta|,$$

and the summation moves over the number of factors: $2 \sim 2k+|\alpha|+|\beta|$.

Lemma 2. *If* $P(x, \xi)$ *satisfies* (2.1) $\sim$ (2.4), *then* $P^*(x, \xi)$ *corresponding the adjoint operator* $P^*(x, D_x) = p_0^*(x, D_x) + p_1^*(x, D_x)$ *satisfies them too for* $p_0^*(x, \xi)$ *and* $p_1^*(x, \xi)$.

Here we construct q_k $k=0, 1, 2, \cdots$, for $P^*(x, \xi)$ by applying (2.5) and (2.6) and we shall use the same notation for q_k in what follows. Setting

$$f_N(x, \xi) = \sum_{k=0}^N q_k$$

and

$$h_N(x, \xi) = p_1 q_N + \sum_{l=0}^N \sum_{|\alpha|+l>N} 1/\alpha!\, P^{(\alpha)} q_{l(\alpha)}$$

we have from (2.5) and (2.6)

$$1 = P^*(x, D_x+\xi) f_N(x, \xi) + h_N(x, \xi).$$

For $\Omega' \subset\subset \Omega$ (relatively compact in Ω) we set $A' = \sup_{x \in \Omega'} A_x$ and choose a function $\psi_0(\xi) \in C_0^\infty(R_\xi^n)$ which equals to 1 in a neighborhood of the set $\{\xi \in R_\xi^n : |\xi| \leqslant A'\}$, and set $\psi_1 = 1 - \psi_0$. As is

$$1 = \psi_1(\xi) + \psi_0(\xi) = P^*(x, D_x+\xi) f_N(x, \xi)\psi_1(\xi) - h_N(x, \xi)\psi_1(\xi) + \psi_0(\xi)$$

we have

(2.7) $$\varphi(x) = P^*(x, D_x)(2\pi)^{-n} \int_{R_\xi^n} e^{i\langle x, \xi\rangle} f_N(x, \xi)\psi_1(\xi)\hat{\varphi}(\xi) d\xi$$
$$-(2\pi)^{-n} \int_{R_\xi^n} e^{i\langle x, \xi\rangle} (h_N(x, \xi)\psi_1(\xi) + \psi_0(\xi))\hat{\varphi}(\xi) d\xi,$$

where $\hat{\varphi}(\xi)$ is the Fourier transform of $\varphi(x) \in C_0^\infty(\Omega)$. For the first term of (2.7) we have

Lemma 3. *The distribution kernel* $F_N(x, y)$ *of the distribution:*

$$\Phi(x, y) \in C_0^\infty(\Omega' \times R^n) \to F_N(\Phi) = (2\pi)^{-n} \int e^{i\langle x, \xi\rangle} f_N(x, \xi)\psi_1(\xi)$$
$$\hat{\Phi}(x, \xi) d\xi dx,$$

where $\hat{\Phi}(x \cdot \xi)$ *denotes the Fourier transform with respect to the second* *variables, is C^∞ function in x and y off the diagonal; $x \neq y$.*

By taking α such that $-m'(1-\rho|\alpha|)<-n$ holds, we have from Lemma 1, (2.2), (2.2)′ and (2.3) that the integral of the right hand side of

$$(x-y)^\alpha F_N(x,y)=(2\pi)^{-n}\int e^{i\langle x-y,\xi\rangle}(-D_\xi)^\alpha(f_N(x,\xi)\psi_1(\xi))d\xi$$

is absolutely convergent at $x \neq y$, from which Lemma 3 is obtained.

For the second term of the right hand side of (2.7) we have

Lemma 4. *The integral of the second term of the right hand side of* (2.7) *is absolutely and uniformly convergent in $C^\kappa(\Omega'\times R^n)$ if $-m'(\rho-\delta)N+\kappa<-n$. And when we set $H_N(x,y)$ the kernel of the integral, we have*

$$\int H_N(x,y)\varphi(y)dy=(2\pi)^{-n}\int e^{i\langle x,\xi\rangle}(h_N(x,\xi)\psi_1(\xi)+\psi_0(\xi))\hat{\varphi}(\xi)d\xi.$$

From the definition of $h_N(x,\xi)$ and (2.2) the integral is estimated by

$$C\,|p_0(x,\xi)|^{-(\rho-\delta)N+\delta\kappa}$$

and by letting N large the exponent becomes negative, by which (2.3) can be used.

By multiplying u a function in $C_0^\infty(\Omega)$ we may assume $u\in\mathcal{E}'(\Omega)$ and hence the order of the distribution u is finite. Let f be in $C^\infty(\omega)$ where ω is a open subset Ω', and $\psi(x)\in C_0^\infty(\Omega')$ be equal to 1 on ω. Here we set

$$g=\psi(x)f, \quad\text{and}\quad h=(1-\psi(x))f.$$

From (2.7) and $P(x,D_x)u=g+h$, we have for $\varphi\in C_0^\infty(\omega)$

$$u(\varphi)=(2\pi)^{-n}\int_{\Omega'}g(x)\left(\int e^{i\langle x,\xi\rangle}f_N(x,\xi)\psi_1(\xi)\hat{\varphi}(\xi)d\xi\right)dx$$
$$+\int_\omega\left(\int_{C\omega}h(x)F_N(x,y)dx\right)\varphi(y)dy+\int_\omega u(H_N(\cdot,y))\varphi(y)dy,$$

where the distribution u operates on $\cdot$ in $H_N(\cdot,y)$. The function $\mathcal{F}_N(x)$ defined by its Fourier transform

$$\hat{\mathcal{F}}_N(\xi)=\int e^{i\langle x,\xi\rangle}f_N(x,\xi)\psi_1(\xi)g(x)dx,$$

is in $C^\infty(\omega)$ by (2.4) and hence we have

$$(2\pi)^{-n}\int\left(\int e^{i\langle x,\xi\rangle}f_N(x,\xi)\psi_1(\xi)g(x)dx\right)\hat{\varphi}(\xi)d\xi$$
$$=\int\mathcal{F}_N(x)\varphi(x)dx.$$

Furthermore by applying Lemma 3 for the second term, and Lemma 4 for the third term of the right hand side of $u(\varphi)$, we can confirm u is smooth of any order in ω.

3. Example.

(1) The symbol $p_0(x,\xi)=|x|^{2\nu}|\xi|^{2\mu}+|\xi|^{2\sigma}+1$, ($\nu>\mu>\sigma\geqq\mu/2$, μ, ν and σ are natural numbers), satisfies the conditions (2.2), (2.3) and (2.4) for $\rho=1/2u$, $\delta=1/2v$ and $m'=2\sigma$.

(2) The symbol $p_0(x,\xi)=\xi_1^4+(x_1^6+x_2^6)(\xi_2^4+\xi_3^4)+\xi_2^2+\xi_3^2$ satisfies the conditions (2.2), (2.3) and (2.4) for $\rho=1/4$, $\delta=1/6$ and $m'=2$.

References

[1] L. Hörmander: Hypoelliptic differential operators. Ann. Inst. Fourier, Grenoble, **11**, 477–492 (1961).

[2] ——: Pseudo-differential operators and hypoelliptic equations. Proc. of Symposia in Pure Math. A.M.S., 138–187 (1966).

[3] В. В. Грушин: Гипоэллиптические Дифференциальные уравнения и псевдо лифференциальные операторы с операторнозначными Символами. Мат. Сбо. **88** (130), 504–521 (1972).

[4] J. Sjöstrand: Une classe d'opérateurs pseudo-différentiels à caractéristiques multiples. C. R. Acad. Sc. Paris, **275** Ser. A, 817–819 (1972).

[5] R. Beals and C. Fefferman: Classes of spatially inhomogeneous pseudo-differential operators (to appear).