57. *An Application of a Certain Argument about Isomorphisms of α-Saturated Structures*

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Applying A. Robinson's proof of the Completeness Theorem, Y. Nakano [3] recently gave a proof of the theorem of Grätzer [2; p. 138, Theorem 6] on the existence of homomorphisms under certain conditions. But, in the theory of models, there is a well known argument about isomorphisms of α-saturated structures (cf. [1; Chap. 11]). As an application of this argument, we shall give a simplified proof of an extended version of the Grätzer's theorem.

We consider each ordinal number as coinciding with the set of smaller ordinal numbers. We use letters ξ, ζ, ρ to denote ordinal numbers and n, i, k to denote natural numbers. We regard cardinals as being identical with initial ordinals. If X is a set we denote its cardinal by $\overline{\overline{X}}$.

Let ρ be an arbitrary ordinal number and let μ be a sequence of natural numbers with domain ρ ($\mu \in \omega^\rho$). By a relational structure of type μ we shall mean a sequence $\mathfrak{A} = \langle A, R_\xi^{\mathfrak{A}} \rangle_{\xi \in \rho}$ where A, the domain of $\mathfrak{A}$, is a non-empty set and $R_\xi^{\mathfrak{A}}$ is a $\mu(\xi)$-ary relation on A for each $\xi < \rho$. Throughout our discussion we shall assume that $\mu \in \omega^\rho$ is some fixed type, that all relational structures we mention are of this type, that L is the appropriate first order language for structures of this type and that for each ordinal ξ, L_ξ is the language obtained from L by adding the ξ-termed sequence of new and distinct constants $\langle c_\zeta : \zeta \in \xi \rangle$. For any relational structure $\mathfrak{A} = \langle A, R_\xi^{\mathfrak{A}} \rangle_{\xi \in \rho}$ and for any ξ-termed sequence $\vec{a} = \langle a_\zeta : \zeta \in \xi \rangle$ of elements of A, we use $(\mathfrak{A}, \vec{a})$ to denote the structure for L_ξ obtained from $\mathfrak{A}$ by interpreting each c_ζ by a_ζ.

Satisfaction of formulas of L_ξ in a structure for L_ξ is defined as usual. If θ is a formula whose free variables are among $v_0, \cdots, v_n$ and if θ holds in $\mathfrak{A}$ with respect to the elements $e_0, \cdots, e_n$ of the domain of $\mathfrak{A}$, then we write $\mathfrak{A} \models \theta[e_0, \cdots, e_n]$.

We use $F(L)$ to designate the set of all formulas of L having at most the one variable v_0 free, and we use $F(L_\xi)$ to designate the corresponding set of formulas of L_ξ.

Suppose Σ is a set of formulas from $F(L_\xi)$, $\mathfrak{A}$ is a relational structure for L and $\vec{a} \in A^\xi$. We say that Σ is simultaneously satisfiable in

$(\mathfrak{A}, \vec{a})$ if there is some $e \in A$ such that for all $\theta \in \Sigma, (\mathfrak{A}, \vec{a}) \models \theta[e]$. Σ is said to be finitely satisfiable in $(\mathfrak{A}, \vec{a})$ if each finite subset of Σ is simultaneously satisfiable in $(\mathfrak{A}, \vec{a})$.

Let α be some cardinal. A relational structure $\mathfrak{A} = \langle A, R_\xi^{\mathfrak{A}} \rangle_{\xi \in \rho}$ is said to be *α-saturated*, if for each ordinal $\xi < \alpha$, and any $\vec{a} \in A^\xi$, a set of formulas which is finitely satisfiable in $(\mathfrak{A}, \vec{a})$ is itself simultaneously satisfiable in $(\mathfrak{A}, \vec{a})$.

Let Γ be a set of formulas. We use $\{\exists, \wedge\}\Gamma$ to designate the set of all formulas that can be formed from the formulas in Γ using only the connective $\wedge$ and quantifier $\exists$.

If $\mathfrak{A}$ is a structure for L_ξ, then we denote by $\mathrm{Th}\,\mathfrak{A}$ the set of all sentences of L_ξ that are valid in $\mathfrak{A}$.

Theorem. *Let $\mathfrak{A} = \langle A, R_\xi^{\mathfrak{A}} \rangle_{\xi \in \rho}$ be a relational structure, and let $\mathfrak{B} = \langle B, R_\xi^{\mathfrak{B}} \rangle_{\xi \in \rho}$ be an $\bar{\bar{A}}$-saturated relational structure. $\mathfrak{A}$ has a homomorphism (an embedding) into $\mathfrak{B}$ if and only if every finite substructure of $\mathfrak{A}$ has a homomorphism (an embedding) into $\mathfrak{B}$.*

Proof. The "only if" part is obvious. To prove the "if" part, assume that every finite substructure of $\mathfrak{A}$ has a homomorphism (an embedding) into $\mathfrak{B}$. We must show that there is a homomorphism (an embedding) from $\mathfrak{A}$ into $\mathfrak{B}$. Let $\vec{a} = \langle a_\xi : \xi < \alpha \rangle$ be an enumeration of A without repetitions. We shall define, by recursion, a sequence $\vec{b} = \langle b_\xi : \xi < \alpha \rangle$ of elements of B such that for all $\xi < \alpha$,

$$\mathrm{Th}\,(\mathfrak{A}, \vec{a} \mid \xi) \cap \{\exists, \wedge\}\Gamma_\xi \subseteq \mathrm{Th}\,(\mathfrak{B}, \vec{b} \mid \xi) \qquad (1)$$

where Γ_ξ is the set of all atomic formulas (all formulas that are either atomic formulas or negations of atomic formulas) of L_ξ.

We must first show that this holds for $\xi = 0$, that is,

$$\mathrm{Th}\,\mathfrak{A} \cap \{\exists, \wedge\}\Gamma_0 \subseteq \mathrm{Th}\,\mathfrak{B}.$$

Let θ be a sentence in $\mathrm{Th}\,\mathfrak{A} \cap \{\exists, \wedge\}\Gamma_0$ and let $(\exists v_1) \cdots (\exists v_n)(\theta_1 \wedge \cdots \wedge \theta_k)$ be a prenex form of θ, where $\theta_i \in \Gamma_0$. Since θ is valid in $\mathfrak{B}$, there are elements $e_1, \cdots, e_n$ of A such that $\mathfrak{A} \models \theta_i[e_1, \cdots, e_n]$ for $i = 1, \cdots, k$. Let $C = \{e_1, \cdots, e_n\}$ and let $\mathfrak{C} = \mathfrak{A} \mid C$ (i.e., $\mathfrak{C}$ is the only substructure of $\mathfrak{A}$ with the domain C). Since $\mathfrak{C}$ is a substructure of $\mathfrak{A}$, $\mathfrak{C} \models \theta_i[e_1, \cdots, e_n]$. By the assumption, $\mathfrak{C}$ has a homomorphism (an embedding) f into $\mathfrak{B}$. Therefore $\mathfrak{B} \models \theta_i[f(e_1), \cdots, f(e_n)]$. Hence it is easily seen that $\theta \in \mathrm{Th}\,\mathfrak{B}$.

Suppose $\xi < \alpha$ and for all $\zeta < \xi$ we have defined b_ζ so that (1) holds. Let $\Sigma = \mathrm{Th}\,(\mathfrak{A}, \vec{a} \mid \xi + 1) \cap \{\exists, \wedge\}\Gamma_{\xi+1}$, and let Σ' be the set of those formulas of $F(L_\xi)$ which are obtained from Σ by replacing all occurrences of the constant c_ξ by the variable v_0. Suppose $\{\theta_1, \cdots, \theta_k\}$ is a finite subset of Σ'. Then $(\mathfrak{A}, \vec{a} \mid \xi) \models \theta_1 \wedge \cdots \wedge \theta_k[a_\xi]$ and so $(\mathfrak{A}, \vec{a} \mid \xi) \models (\exists v_0)(\theta_1 \wedge \cdots \wedge \theta_k)$. Hence, by the hypothesis (1), $(\mathfrak{B}, \vec{b} \mid \xi) \models (\exists v_0)(\theta_1 \wedge \cdots \wedge \theta_k)$, and so there is some $e \in B$ such that $(\mathfrak{B}, \vec{b} \mid \xi) \models \theta_i[e]$ for $i = 1, \cdots, k$. Therefore we have shown that Σ' is finitely satisfiable in

$(\mathfrak{B}, \vec{b}\,|\,\xi)$. But $\mathfrak{B}$ is α-saturated and therefore we may choose $b_\xi \in B$ so that for all $\theta \in \Sigma'$, $(\mathfrak{B}, b\,|\,\xi) \models \theta[b_\xi]$. It follows that

$$\mathrm{Th}\,(\mathfrak{A}, \vec{a}\,|\,\xi+1) \cap \{\exists, \wedge\}\Gamma_{\xi+1} \subseteq \mathrm{Th}\,(\mathfrak{B}, \vec{b}\,|\,\xi+1).$$

This completes the recursive definition of $\vec{b}$.

Clearly, by (1), $\mathrm{Th}\,(\mathfrak{A}, \vec{a}) \cap \{\exists, \wedge\}\Gamma_\alpha \subseteq \mathrm{Th}\,(\mathfrak{B}, \vec{b})$. Therefore, if we define g by $g(a_\xi) = b_\xi, \xi < \alpha$, then we can easily see that g is a homomorphism (an embedding) of $\mathfrak{A}$ into $\mathfrak{B}$. q.e.d.

The following is a version of the Grätzer's theorem (cf. [3]).

Corollary. *Let $\mathfrak{B}$ be a finite relational structure. A relational structure $\mathfrak{A}$ has a homomorphism into $\mathfrak{B}$ if and only if every finite substructure of $\mathfrak{A}$ has a homomorphism into $\mathfrak{B}$.*

Proof. By the simple fact that finite structure $\mathfrak{B}$ is α-saturated for each cardinal α (cf. [1; p. 218]), the result follows immediately from the theorem.

References

[1] J. L. Bell and A. B. Slomson: Models and Ultraproducts. North-Holland (1969).
[2] G. Grätzer: Universal Algebra. Van Nostrand (1968).
[3] Y. Nakano: An application of A. Robinson's proof of the completeness theorem. Proc. Japan Acad., **47**, 929–931 (1971).