55. A Remark on the Normal Expectations. II

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(Comm. by Kinjirô KUNUGI, M. J. A., April 12, 1973)

1. In the previous note [3], the concept of generalized channels is introduced. In the note [2], it is proved that, for a von Neumann algebra and a von Neumann subalgebra of it, the conjugate mapping of a generalized channel with a certain property is a normal expectation.

In this note, we shall show that a generalized channel is considered a normal expectation.

2. Consider a von Neumann algebra $\mathcal{A}$, denote the conjugate space of $\mathcal{A}$ as $\mathcal{A}^*$ and the subconjugate space of all ultra-weakly continuous linear functionals on $\mathcal{A}$ as $\mathcal{A}_*$, following after the definition of Dixmier [4].

Definition (cf. [3]). Let $\mathcal{A}$ and $\mathcal{B}$ be two von Neumann algebras, then a positive linear mapping π of $\mathcal{A}_*$ into $\mathcal{B}_*$ is called a *generalized channel* if π maps a normal state to a normal state.

The following proposition is obtained in [3]:

Proposition 1. *A positive linear mapping π of $\mathcal{A}_*$ into $\mathcal{B}_*$ is a generalized channel if and only if the conjugate mapping π^* is a positive normal linear mapping of $\mathcal{B}$ into $\mathcal{A}$ preserving the identity.*

In the sequel, according to this proposition, a normal positive linear mapping of a von Neumann algebra into a von Neumann algebra preserving the identity will be called also a generalized channel.

Let $\mathcal{A}$ be a von Neumann algebra and $\mathcal{B}$ a von Neumann subalgebra of $\mathcal{A}$, then a positive linear mapping e of $\mathcal{A}$ onto $\mathcal{B}$ is called an *expectation* of $\mathcal{A}$ onto $\mathcal{B}$ if e satisfies the following conditions:

(i) $1^e=1$, and

(ii) $(BAC)^e=BA^eC$ for all $A\in\mathcal{A}$ and $B,C\in\mathcal{B}$, cf. [5].

The following proposition is proved in [2]:

Proposition 2. *Let $\mathcal{A}$ be a von Neumann algebra and $\mathcal{B}$ a von Neumann subalgebra of $\mathcal{A}$, then a mapping π of $\mathcal{B}_*$ to $\mathcal{A}_*$ is a generalized channel with*

(1) $$\pi L_B=L_B\pi \qquad \text{for any } B\in\mathcal{B}$$

if and only if the conjugate mapping e of $\mathcal{A}$ onto $\mathcal{B}$ is a normal expectation, where a mapping L_A on $\mathcal{A}^*$ is defined for $A\in\mathcal{A}$ by

(2) $$L_Af(X)=f(AX) \qquad \text{for all } f\in\mathcal{A}^* \text{ and } X\in\mathcal{A}.$$

Let $\mathcal{A}\otimes\mathcal{B}$ be the tensor product of von Neumann algebras $\mathcal{A}$ and

$\mathcal{B}$. We shall identify $\mathcal{A}\otimes 1$ (resp. $1\otimes\mathcal{B}$) with $\mathcal{A}$ (resp. $\mathcal{B}$).

Theorem 1. *Let $\mathcal{A}\otimes\mathcal{B}$ be the tensor product of von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$. Let π be a generalized channel of $\mathcal{A}$ (resp. $\mathcal{B}$) to $\mathcal{B}$ (resp. $\mathcal{A}$) and ψ (resp. φ) a normal state on $\mathcal{A}$ (resp. $\mathcal{B}$). Then there exists a normal expectation e (resp. ε) of $\mathcal{A}\otimes\mathcal{B}$ to $\mathcal{B}$ (resp. $\mathcal{A}$) such that*

$$e(A\otimes B)=\varphi(\pi(A))B \text{ (resp. } \varepsilon\,(A\otimes B)=\psi(\pi(B))A)$$

for $A\in\mathcal{A}$ and $B\in\mathcal{B}$.

Proof. Denote by $g\otimes f$ a ultra-weakly continuous linear functional on $\mathcal{A}\otimes\mathcal{B}$ for $g\in\mathcal{A}_*$ and $f\in\mathcal{B}_*$ such that

$$g\otimes f(A\otimes B)=g(A)f(B) \qquad \text{for } A\in\mathcal{A} \text{ and } B\in\mathcal{B},$$

cf. [2; p. 64]. Since $\varphi\circ\pi$ is a normal state, we can define a mapping e_* of $\mathcal{B}_*$ to $(\mathcal{A}\otimes\mathcal{B})_*$ by the following:

$$e_*(f)=\varphi\circ\pi\otimes f \qquad \text{for every } f\in\mathcal{B}_*.$$

If f is a normal state on $\mathcal{B}$, then $e_*(f)$ is also. It implies that e_* is a generalized channel of $\mathcal{B}_*$ to $(\mathcal{A}\otimes\mathcal{B})_*$. It is clear that

$$e_*L_B(f)(A\otimes C)=\varphi\circ\pi(A)f(BC)=L_Be_*(f)(A\otimes C)$$

for $f\in\mathcal{B}_*$, $A\in\mathcal{A}$ and $B,C\in\mathcal{B}$. Therefore, by Proposition 2, the conjugate mapping e of e_* is a normal expectation of $\mathcal{A}\otimes\mathcal{B}$ onto $\mathcal{B}$. By the definition of e, it is clear that e satisfies

$$e(A\otimes B)=\varphi(\pi(A))B \qquad \text{for } A\in\mathcal{A} \text{ and } B\in\mathcal{B}.$$

The argument for ε goes similarly.

Take and fix a normal state φ on $\mathcal{A}$. Put $\pi(A)=\varphi(A)\cdot 1$ for each $A\in\mathcal{A}$ and the identity 1 of $\mathcal{B}$, then π is a generalized channel of $\mathcal{A}$ to $\mathcal{B}$. Therefore the theorem implies the following

Corollary 2. *Let $\mathcal{A}\otimes\mathcal{B}$ be the tensor product of von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$. Then each normal state φ on $\mathcal{A}$ (resp. ψ on $\mathcal{B}$) induces a normal expectation e of $\mathcal{A}\otimes\mathcal{B}$ onto $\mathcal{B}$ (resp. $\mathcal{A}$) such that*

$$e(A\otimes B)=\varphi(A)B \qquad (\textit{resp. } e(A\otimes B)=\psi(B)A).$$

3. In this section, we shall discuss a completely positive linear mapping. A positive linear mapping π of a von Neumann algebra $\mathcal{A}$ to a von Neumann algebra $\mathcal{B}$ is called *completely positive* in the sense of Stinespring [6] (*positive definite* in the sense of Umegaki [7]), if $\pi(n)$, defined by

$$\pi(n)(A_{ij})=(\pi(A_{ij})),$$

is positive on the $n\times n$ matrix algebra over $\mathcal{A}$, for every n.

Lemma 3. *Let $\mathcal{A}$ be a von Neumann algebra, $\mathcal{B}$ an abelian von Neumann algebra and π a completely positive linear mapping of $\mathcal{A}$ to $\mathcal{B}$ preserving the identity. Then every state ψ on $\mathcal{B}$ induces a state φ on $\mathcal{A}\otimes\mathcal{B}$ such that*

$$\varphi(A\otimes B)=\psi(\pi(A)B)$$

for every $A\in\mathcal{A}$ and $B\in\mathcal{B}$.

Proof. Let $\mathcal{A}\odot\mathcal{B}$ be the algebraic tensor product of $\mathcal{A}$ and $\mathcal{B}$, and ψ a state on $\mathcal{B}$. Put

$$\varphi\left(\sum_{i=1}^{n} A_i\otimes B_i\right)=\sum_{i=1}^{n} \psi(\pi(A_i)B_i),$$

for all

$$\sum_{i=1}^{n} A_i\otimes B_i \in \mathcal{A}\odot\mathcal{B}.$$

By the definition of the algebraic tensor product, φ is well-defined.

Let π_ψ be the representation of $\mathcal{B}$ on a Hilbert space $\mathfrak{H}_\psi$ induced by a positive linear functional ψ of $\mathcal{B}$. Put

$$U=\sum_{i=1}^{n} A_i\otimes B_i \in \mathcal{A}\odot\mathcal{B}.$$

Let $x \in \mathfrak{H}_\psi$ be the cyclic vector with

$$\psi(T)=(\pi_\psi(T)x, x)$$

for any $T \in \mathcal{B}$. Then we have

$$\begin{aligned}\varphi(U^*U)&=\sum_{i,j} \psi(\pi(A_i^*A_j)B_i^*B_j)\\ &=\sum_{i,j} (\pi_\psi(\pi(A_i^*A_j)B_i^*B_j)x, x)\\ &=\sum_{i,j} (\pi_\psi(\pi(A_i^*A_j))\pi_\psi(B_j)x, \pi_\psi(B_i)x).\end{aligned}$$

Since π_ψ and π are completely positive, the composition $\pi_\psi\circ\pi$ is completely positive too, and so φ is positive. It is clear that φ is linear and $\varphi(1)=1$. Therefore φ is a state on $\mathcal{A}\odot\mathcal{B}$, and there exists a state on $\mathcal{A}\otimes\mathcal{B}$ which is the extension of φ. Denote by the same notation φ the extension, then we have a state φ on $\mathcal{A}\otimes\mathcal{B}$ such that

$$\varphi(A\otimes B)=\psi(\pi(A)B) \qquad \text{for every } A \in \mathcal{A} \text{ and } B \in \mathcal{B}.$$

Remark. By a theorem of Arveson [1; Proposition 1.2.2], the hypothesis of the complete positivity of the mapping is reduced to the positivity since every positive linear mapping of a C^*-algebra into an abelian C^*-algebra is automatically completely positive.

Let $\mathcal{B}$ be a von Neumann subalgebra of a von Neumann algebra $\mathcal{A}$. Following after the definition of Umegaki [8], a normal state φ on $\mathcal{A}$ is called a *$\mathcal{B}$-tracelet* if φ satisfies

(3) $\qquad \varphi(AB)=\varphi(BA) \qquad$ for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Umegaki proved in [8]:

Theorem A. *Let $\mathcal{A}$ be a von Neumann algebra and $\mathcal{B}$ a von Neumann subalgebra of $\mathcal{A}$. For any faithful $\mathcal{B}$-tracelet φ, there exists a normal expectation e of $\mathcal{A}$ onto $\mathcal{B}$ such that*

(4) $\qquad \varphi(A)=\varphi(e(A)) \qquad$ *for every $A \in \mathcal{A}$.*

Now, we shall show the following theorem:

Theorem 4. *Let $\mathcal{A}$ be a von Neumann algebra, $\mathcal{B}$ a σ-finite abelian von Neumann algebra and π a positive linear mapping of $\mathcal{A}$ to $\mathcal{B}$ preserving the identity. Then there exists an expectation e of $\mathcal{A}\otimes\mathcal{B}$ onto $1\otimes\mathcal{B}$ such that $e(A\otimes 1)=1\otimes\pi(A)$.*

Proof. Let ψ be a faithful normal state on $\mathcal{B}$. By Lemma 3, there exists a state φ on $\mathcal{A}\otimes\mathcal{B}$ such that

$$\varphi(A\otimes B)=\psi(\pi(A)B) \qquad \text{for every } A\in\mathcal{A} \text{ and } B\in\mathcal{B}.$$

Denote by σ the vector state on $\pi_\varphi(\mathcal{A}\otimes\mathcal{B})$ induced by φ, that is, σ is the faithful normal state on $\pi_\varphi(\mathcal{A}\otimes\mathcal{B})$ with $\sigma(\pi_\varphi(T))=\varphi(T)$. Since $\mathcal{B}$ is abelian, $1\otimes\mathcal{B}$ is contained in the center of $\mathcal{A}\otimes\mathcal{B}$, so $\pi_\varphi(1\otimes\mathcal{B})$ is contained in the center of $\pi_\varphi(\mathcal{A}\otimes\mathcal{B})$. Therefore, σ is a faithful $\pi_\varphi(1\otimes\mathcal{B})$-tracelet. By Theorem A, there exists a normal expectation ε of $\pi_\varphi(\mathcal{A}\otimes\mathcal{B})$ onto $\pi_\varphi(1\otimes\mathcal{B})$. On the other hand, φ is faithful on $1\otimes\mathcal{B}$ by the property of ψ, so π_φ is an isomorphism of $1\otimes\mathcal{B}$ onto $\pi_\varphi(1\otimes\mathcal{B})$. Let π_φ^{-1} be the inverse of π_φ of $\pi_\varphi(1\otimes\mathcal{B})$ onto $1\otimes\mathcal{B}$. Put $e=\pi_\varphi^{-1}\circ\varepsilon\circ\pi_\varphi$, then e is an expectation of $\mathcal{A}\otimes\mathcal{B}$ onto $1\otimes\mathcal{B}$. By (4) of ε and the definition of e, we have the following equalities:

$$\begin{aligned}\varphi(e(A\otimes 1)(1\otimes B)) &= \sigma(\varepsilon\cdot\pi_\varphi(A\otimes 1)\pi_\varphi(1\otimes B))\\ &= \sigma(\pi_\varphi(A\otimes 1)\pi_\varphi(1\otimes B))\\ &= \varphi((A\otimes 1)(1\otimes B))\\ &= \varphi(A\otimes B)\\ &= \psi(\pi(A)B)\\ &= \varphi((1\otimes\pi(A))(1\otimes B)),\end{aligned}$$

for every $A\in\mathcal{A}$ and $B\in\mathcal{B}$. Therefore

$$e(A\otimes 1)=1\otimes\pi(A) \qquad \text{for every } A\in\mathcal{A},$$

because φ is faithful on $1\otimes\mathcal{B}$, which completes the proof.

Corollary 5. *Let $\mathcal{A}$ be a von Neumann algebra, $\mathcal{B}$ a σ-finite abelian von Neumann algebra and π a generalized channel of $\mathcal{A}$ to $\mathcal{B}$. Then there exists an expectation e of $\mathcal{A}\otimes\mathcal{B}$ onto $1\otimes\mathcal{B}$ with*

$$e(A\otimes 1)=1\otimes\pi(A)$$

for all $A\in\mathcal{A}$.

By this corollary, a generalized channel of a von Neumann algebra to a σ-finite abelian von Neumann algebra is considered an expectation. Furthermore, the proof of Theorem 4 tells us that the generalized channel is considered a normal expectation.

References

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