51. *On Some Hyperbolic Equations with Operator Coefficients*

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1. We consider first the class of singular Cauchy problems for $u^m \in C^2(E)$ on $[0, T]$.

(1.1) $$u^m_{tt}+(2m+1)\coth t\, u^m_t+m(m+1)u^m=A^2u^m$$

(1.2) $$u^m(0)=u_0\,; \qquad u^m_t(0)=0$$

where $u_0 \in E$ is given and A is the generator of a locally equicontinuous group $T(t)=\exp At$ in a complete locally convex Hausdorff space E (cf. [9]). When A^2 is replaced by the Laplace-Beltrami operator Δ in function spaces E over $M=SL(2, R)/SO(2)$ and $m\geqslant 0$ is an integer, these equations arise in a canonical way from certain Lie group theoretic considerations and are parallel to the corresponding Euler-Poisson-Darboux (EPD) equations (cf. [2], [4], [10]); in fact there are many parallel theories for canonical classes of singular Cauchy problems but we will only deal here with (1.1)–(1.2) (cf. [5], [10] for other situations).

Now there are two canonical recursion relations arising from the group theory when A^2 is replaced by Δ which we write in the form

(1.3) $$u^m_t+2m\coth t\, u^m=2m\,\mathrm{csch}\, t\, u^{m-1}$$

(1.4) $$u^m_t=\frac{\sinh t}{2(m+1)}[A^2-m(m+1)]u^{m+1}$$

and (1.3) leads directly to a generalized Sonine formula (sh = sinh and ch = cosh)

(1.5) $$\mathrm{sh}^{2m}\, t\, u^m(t)=c(m, l)\int_0^t(\mathrm{ch}\, t-\mathrm{ch}\, y)^{l-1}\, \mathrm{sh}^{2m-2l+1}\, y\, u^{m-l}(y)dy$$

where $c(m, l)=2^l\Gamma(m+1)/\Gamma(m-l+1)\Gamma(l)$ and (temporarily) $m\geqslant l\geqslant 1$ are integers. Thus, for example, when $m=l\geqslant 1$ is an integer one connects u^m to the mean value solution u^0 and (1.4)–(1.5) yield a growth theorem $u^m_t\geqslant 0$ for $m\geqslant 0$ whenevr $[\Delta-m(m+1)]u_0\geqslant 0$ (since $\Delta u^0(t, u_0)=u^0(t, \Delta u_0)$). This and similar convexity theorems (see [2]; [4]; [10]) are parallel to those of Weinstein [11] for EPD equations with $m\geqslant 0$ arbitrary (cf. also [1], [7], [12]). The Weinstein recursion relations for the EPD theory correspond to a version of (1.4) plus a relation connecting u^m to u^{-m} (see remarks after (3.1)); a parallel form of a version of (1.3) was also known. The (1.3)–(1.4) analogues were however first systematically exploited together in existence-uniqueness theory for EPD

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equations in [1] and their group theoretic significance was discovered in [2], [4], [10]. The Sonine formulas are also Riemann-Liouville integrals and are connected with the transmutation operators of Delsarte-Lions.

We give here some new types of growth and convexity theorems for (1.1)–(1.2) when E is a suitable space of functions and $m\geqslant -1/2$ is arbitrary (see [3], [5] for details); theorems of the type mentioned above can also be extended to arbitrary $m\geqslant 0$ but we will omit this (cf. [5]). We develop existence and uniqueness theorems for general parameter values with general E and extend the recursion relations. Also some general existence-uniqueness theorems for differential equations with constant operator coefficients are indicated in section 4. The constructions are based on a technique of Hersh [8] for Banach spaces and we remark that the context of more general locally convex E and locally equicontinuous groups is literally forced upon us by the growth phenomena involved.

2. Let $l(l+1)=-s$ ($s\in R$) with $l=-1/2+iv$ where $v=(s^2-1/4)^{1/2}$ and set $z=\operatorname{ch} t$ so that $1\leqslant z<\infty$. Then define for m not a negative integer

$$\hat{R}^m(t,s)=\frac{2\Gamma(m+1)}{(z^2-1)^{m/2}}P_l^{-m}(z) \tag{2.1}$$

where $P_l^{-m}(z)$ denotes the associated Legendre function of the first kind. The negative integers seem to be exceptional as in EPD theory. One thinks of $\hat{R}^m(t,s)=\mathcal{F}R_x^m(t)$ where $\mathcal{F}$ denotes the Fourier transform ($x\to s$) and using the relation (P_l is the Legendre function defined for any $l\in C$)

$$P_l^{-m}(z)=(z^2-1)^{-m/2}(J^k(J^{m-k}P_l))(z) \tag{2.2}$$

with $m-k=-1/2$ and $\operatorname{Re} m>-1/2$ (J^k denotes the Riemann-Liouville fractional integration from 1 to z) we have

Theorem 2.1. *Let $R_x^m(\cdot)\in C^\infty(\mathcal{E}'_x)$ for $\operatorname{Re} m>-1/2$ be defined by*

$$\operatorname{sh}^{2m} t\, R_x^m(t)=c\left(m,m+\frac{1}{2}\right)\int_0^t(\operatorname{ch} t-\operatorname{ch} y)^{m-1/2}R_x^{-1/2}(y)dy \tag{2.3}$$

where $R_x^{-1/2}(y)=\mu_x(y)+(1/4)y\int_0^y K(y,w)\mu_x(w)dw$ is the unique solution in $C^2(\mathcal{E}'_x)$ of the telegraph equation (2.6) with $m=-1/2$ satisfying (2.7) (with $\mu_x(y)=1/2[\delta(x+y)+\delta(x-y)]$ and $K(y,w)=J_1(ia)/ia\geqslant 0$ where $a=(y^2-w^2)^{1/2}/2$). Then $\mathcal{F}R_x^m(t)=\hat{R}^m(t,s)$ and setting $D=\partial/\partial t$ with $\Delta=\partial^2/\partial x^2$

$$DR_x^m(t)=\frac{\operatorname{sh} t}{2(m+1)}[\Delta-m(m+1)]R_x^{m+1}(t) \tag{2.4}$$

$$DR_x^m(t)+2m\coth t\, R_x^m(t)=2m\operatorname{csch} t\, R_x^{m-1}(t) \tag{2.5}$$

$$D^2R_x^m(t)+(2m+1)\coth t\, DR_x^m(t)+m(m+1)R_x^m(t)=\Delta R_x^m(t) \tag{2.6}$$

$$R_x^m(0)=\delta(x);\ DR_x^m(0)=0 \tag{2.7}$$

We define then for $\operatorname{Re} m>-1/2$ (or $m=-1/2$) $u^m(t)$

$=\langle R_x^m(t), T(x)u_0\rangle$ where it is assumed that $u_0 \in D(A^2)$. This can be written out in the form (cf. (2.3))

$$\text{sh}^{2m}\, t\, u^m(t) = \int_0^t I_m(t, y)\langle R_x^{-1/2}(y), T(x)u_0\rangle dy \tag{2.8}$$

where $I_m(t, y) = c(m, m+1/2)(\text{ch}\, t - \text{ch}\, y)^{m-1/2}$ and, setting $\text{ch}\, At = 2^{-1}[T(t)+T(-t)]$,

$$\langle R_x^{-1/2}(y), T(x)u_0\rangle = \text{ch}\, Ay\, u_0 + \frac{1}{4}y\int_0^y K(y, w)\, \text{ch}\, Aw\, u_0\, dw \tag{2.9}$$

Theorem 2.2. *Let A generate a locally equicontinuous group in a complete locally convex Hausdorff space E. Assume $u_0 \in D(A^2)$ and define $u^m(t)$ for* $\text{Re}\, m > -1/2$ *(or* $m = -1/2$*) by* (2.8)–(2.9). *Then $u^m \in C^2(E)$ satisfies* (1.1)–(1.2) *and the recursion relations* (1.3)–(1.4) *hold.*

We write now $A_m^2 = A^2 - m(m+1)$ and combine (1.4) with (2.8)–(2.9) for a growth theorem; for convexity we write (1.1) in the form $(d^2/df^2)u^m(t) = \text{sh}^{4m+2}\, t\, A_m^2 u^m(t)$ where $df/dt = \text{csch}^{2m+1}\, t$ (cf. [1]–[5], [7], [10]–[12]). There results

Theorem 2.3. *Let u^m with $m \geqslant -1/2$ real be constructed as in Theorem* 2.2 *where E is a space of functions. Then if* $\text{ch}\, At(A_m^2 u_0) \geqslant 0$ *for $0 \leqslant t \leqslant t_0$ it follows that u^m is monotone nondecreasing on* $[0, t_0]$ *and aconvex function of f.*

Remark 2.4. Various nontrivial concrete examples of E and A are given in [3], [5] where realizable conditions on u_0 will imply $\text{ch}\, At(A_m^2 u_0) \geqslant 0$. It is here that one is led to work in "large" spaces and use locally equicontinuous groups.

3. We will indicate now two types of uniqueness theorems in the context of Theorem 2.2 and along the way some existence theorems for (1.1) with other indices will be established (see again [3], [5] for details).

Theorem 3.1. *Let A generate a locally equicontinuous group in E where E' is complete and $D(A^*) \subset E'$ is dense. Then the solutions to* (1.1)–(1.2) *constructed in Theorem* 2.2 *with* $\text{Re}\, m > -1/2$ *or* $m = -1/2$ *are unique if m is not an integer.*

The proof of Theorem 3.1 uses the following result

Theorem 3.2. *Under the hypotheses of Theorem* 3.1 *let $u_0^* \in D((A^*)^2)$ be arbitrary and $0 < \hat{t} \leqslant T$. Let $n = -m-1$ for* $\text{Re}\, m > -1/2$ *or $m = -1/2$ with m not an integer. Then there exists a solution $Y_x^n(t)$ of* (2.6) *with index n satisfying $Y_x^n(\hat{t}) = 0$ and $DY_x^n(\hat{t}) = \delta(x)$ $(D = \partial/\partial t)$. If $T^*(t)$ is the locally equicontinuous group generated by A^* then $p^n(t) = \langle Y_x^n(t), T^*(x)u_0^*\rangle$ satisfies* (1.1) *with index n and A replaced by A^* while $p^n(\hat{t}) = 0$ and $p_t^n(\hat{t}) = u_0^*$.*

Thus, given u^m a solution of (1.1)–(1.2) with $u_0 = 0$, one brackets (1.1) with p_t^n, where p^n is obtained from Theorem 3.2, and integrating by parts we obtain $\langle u_t^m(\hat{t}), u_0^*\rangle = 0$ from which follows $u^m(t) \equiv 0$ on $[0, T]$

since $\hat{t}$ is arbitrary with $D((A^*)^2)$ dense (note here that $-(2m+1)=2n+1$ with $m(m+1)=n(n+1)$). In order to obtain $Y_x^n(t)$ as in Theorem 3.2 we first embed the $R_x^m(t)$ of Theorem 2.1 in an extended family by means of the Weinstein type formula

$$R_x^m(t)=\frac{(z^2-1)^{-m}(\partial/\partial z)^p}{2^p(m+p)\cdots(m+1)}[(z^2-1)^{m+p}R_x^{m+p}(t)] \tag{3.1}$$

where $z=\mathrm{ch}\,t, p$ is an integer chosen so that $\mathrm{Re}\,(m+p)>-1/2$ (or $m+p=-1/2$), and m is not a negative integer; the recursion relations (2.4)–(2.5) remain valid in the extended family. In deriving (3.1) one uses the fact that $\mathrm{sh}^{-2m}\,t\,R_x^{-m}(t)$ satisfies (2.6) with index m provided $R_x^{-m}(t)$ satisfies (2.6) with index $-m$. Now since $P_l^m(z)$ and $P_l^{-m}(z)$ are linearly independent solutions of the associated Legendre equation for m not an integer a second solution of $(2.6)^\wedge=\mathcal{F}$ (2.6) ($\Delta\to-s^2$) is given by $\hat{W}^m(t,s)=2^m\Gamma(m+1)(z^2-1)^{-m/2}P_l^m(z)$ (cf. (2.1)). Using (2.2) again one defines

$$W_x^n(t)=b_m\int_0^t(\mathrm{ch}\,t-\mathrm{ch}\,y)^{m-1/2}\,\mathrm{sh}\,y\,W_x^{-1/2}(y)dy \tag{3.2}$$

(cf. (2.3)) where $b_m=2^{-m-1/2}\Gamma(-m)/\Gamma(1/2)\Gamma(m+1/2)$, $n=-m-1$ with $\mathrm{Re}\,m>-1/2$ (m not an integer), and with $\mu_k(w)$ as before, $W_x^{-1/2}(t)=\exp(-(1/2)t)\int_0^t J_0((1/2)i(t^2-w^2)^{1/2})\mu_x(w)dw$ so that $W_x^{-1/2}(0)=0$ and $DW_x^{-1/2}(0)=\delta(x)$. Finally for $\mathrm{Re}\,n<-1/2$ or $n=-1/2$ with n not an integer $Y_x^n(t)$ can be constructed as a suitable linear combination of $R_x^n(t)$ and $W_x^n(t)$. The limitation that m not be an integer in Theorem 3.1 can probably be removed but we have not yet investigated this carefully (see [5]).

In order to have a uniqueness theorem without the limitation that E' be complete with $D(A^*)$ dense we will indicate a procedure (see [3]) based on properties of the Riemann-Liouville integral (a similar method for the EPD equation in a Banach space is suggested by Donaldson in [6]). This method involves additional smoothness hypotheses on the solution however (as in [6]) but it seems that a suitable distribution argument using the same formalism will eliminate the need for extra regularity. We will sketch the formalism here in the smooth case treated in [3] and leave the refinements for [5]. Thus let $u^m(t)=w^m(z)$ ($z=\mathrm{ch}\,t$) for $\mathrm{Re}\,m>-1/2$ satisfy (1.1)–(1.2) with $u_0=0$ and set

$$F^m(z)=\frac{(z^2-1)^m w^m(z)}{\Gamma(m+1)2^m} \tag{3.3}$$

We consider the function $F^{-1/2}$ *defined* by $F^{-1/2}=J^{-m-1/2}F^m$. Given that F^m will satisfy a certain differential equation $P_m(z,D_z,A)F^m=0$ (since u^m satisfies (1.1)) it follows that, under suitable regularity of F^m, $P_{-1/2}(z,D_z,A)F^{-1/2}=0$ and the $u^{-1/2}$ arising from $F^{-1/2}$ via (3.3) will then satisfy the generalized telegraph equation (1.1) with $m=-1/2$ where

$u^{-1/2}(0)=u_t^{-1/2}(0)=0$. Hence if such $u^{-1/2}(t)$ are necessarily zero and if $m+1/2$ is not an integer, so that one recovers F^m by the formula $F^m=J^{m+1/2}F^{-1/2}$, it follows that $u^m(t)\equiv 0$. These results can be achieved for example if $F^m\in C^{b+3}(E)$, $F^m(1)=F_z^m(1)=0$, and $A^2u_t^{-1/2}\in C^0(E)$ where $b-1/2\leqslant \mathrm{Re}\, m<b+1/2$ with $b\geqslant 0$ an integer or $m=b-1/2$ ($\mathrm{Re}\, m=-1/2$ is excluded).

4. Some general existence-uniqueness theorems are also proved in [5] for general hyperbolic differential equations with operator coefficients in E of the form

$$P(D, A)u=\sum_{j=0}^{m} P_j(A)D^j u=\sum_{j=0}^{m}\sum_{i=0}^{p} c_{j,i}A^i D^j u=0 \tag{4.1}$$

where E is a complete locally convex Hausdorff space in which A generates a locally equicontinuous group $T(t)$, $D=\partial/\partial t$, $0\leqslant t\leqslant T<\infty$, the $c_{j,i}$ are constant, and (for convenience) $P_m(A)=1$ (i.e., $c_{m,0}=1$ with $c_{m,i}=0$ for $i>0$). A solution to (4.1) satisfying $u^k(0)=u_k$ for $0\leqslant k\leqslant m-1$ with suitable u_k is constructed (following Hersh [8]) in the form

$$u(t)=\sum_{k=0}^{m-1}\langle g_k(t,\cdot), T(\cdot)u_k\rangle \tag{4.2}$$

where $P(D, -D_x)$ is hyperbolic in the sense of Gelfand-Šilov and

$$P(D, -D_x)g_k=0\,;\ D^l g_k|_{t=0}=\delta_{l,k}\delta(x) \tag{4.3}$$

where $\delta_{l,k}$ is the Kronecker symbol ($0\leqslant l\leqslant m-1$). The technique is essentially just an extension of that given in [8] and it was mainly developed here in this general context in order to provide a framework in which to study growth theorems for various equations using operational calculus (also more specific hypotheses on the initial data are given). The material on growth theorems is still in preparation and most of it will appear in [5]; the results already indicated in section 2 will appear in [3] and various other cases of time dependent coefficients will also be treated in [5].

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