69. *Further Results for the Solutions of Certain Third Order Non-autonomous Differential Equations*

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1. Introduction. The differential equations considered here are

(1.1) $$\dddot{x}+\psi(t,x,\dot{x},\ddot{x})+\phi(t,x,\dot{x})+c(t)f(x)=p(t,x,\dot{x},\ddot{x})$$

(1.2) $$\dddot{x}+\psi(t,x,\dot{x},\ddot{x})+\phi(t,x,\dot{x})+c(t)f(x)=0$$

where ψ, ϕ, c, f and p are real valued functions. All solutions of (1.1) considered here are assumed real.

In [4] M. Harrow considered the behavior as $t\to\infty$ of solutions of the differential equation

(1.3) $$\dddot{x}+f(x,\dot{x},\ddot{x})\ddot{x}+g(x,\dot{x})+h(x)=p(t).$$

In [6] H. O. Tejumola considered the behavior as $t\to\infty$ of solutions of the differential equation

(1.4) $$\dddot{x}+f(t,\dot{x},\ddot{x})\ddot{x}+g(x,\dot{x})+h(x)=p(t,x,\dot{x},\ddot{x}).$$

Recently, in [3] T. Hara obtained some conditions under which all solutions of the equation

(1.5) $$\dddot{x}+a(t)f(x,\dot{x},\ddot{x})\ddot{x}+b(t)g(x,\dot{x})+c(t)h(x)=p(t,x,\dot{x},\ddot{x})$$

tend to zero as $t\to\infty$.

In [7], the author established conditions under which all solutions of the non-autonomous equation (1.1) tend to zero as $t\to\infty$.

In this note we investigate the asymptotic behavior of the solutions of the equation (1.1) under the condition weaker than that obtained in [3], [4], [6].

Many results have been obtained on the asymptotic properties of autonomous equations of third order and many of these results are summarized in [5].

2. Assumptions and Theorems. We shall state the assumptions on the functions ψ, ϕ, f, c and p appeared in the equation (1.1).

Assumptions.

(I) *$f(x)$ is a C^1-function in R^1, and $c(t)$ is a C^1-function in $I=[0,\infty)$.*

(II) *The function $\phi(t,x,y)$ is continuous in $I\times R^2$, and for the function $\phi(t,x,y)$ there exist functions $b(t), \phi_0(x,y)$ and $\phi_1(x,y)$ which satisfy the inequality*
$$b(t)\phi_0(x,y)\leqq\phi(t,x,y)\leqq b(t)\phi_1(x,y) \qquad \text{in } I\times R^2.$$
Moreover $b(t)$ is a C^1-function in I.

Let $\tilde{\phi}(x,y)\equiv\frac{1}{2}\{\phi_0(x,y)+\phi_1(x,y)\}$, $\tilde{\phi}(x,y)$ *and* $\frac{\partial\tilde{\phi}}{\partial x}(x,y)$ *are continuous in* R^2.

(III) *The function* $\psi(t,x,y,z)$ *is continuous in* $I\times R^3$, *and for the function* $\psi(t,x,y,z)$ *there exist functions* $a(t)$, $\psi_0(x,y,z)$ *and* $\psi_1(x,y,z)$ *which satisfy*

$$a(t)\psi_0(x,y,z)\leqq\frac{1}{z}\psi(t,x,y,z)\leqq a(t)\psi_1(x,y,z)\qquad in\ I\times R^3.$$

Further $a(t)$ *is a* C^1*-function in* I, *and let*

$$\tilde{\psi}(x,y,z)\equiv\frac{1}{2}\{\psi_0(x,y,z)+\psi_1(x,y,z)\},$$

$\tilde{\psi}(x,y,z)$, $\frac{\partial\tilde{\psi}}{\partial x}(x,y,z)$ *and* $\frac{\partial\tilde{\psi}}{\partial z}(x,y,z)$ *are continuous in* R^3.

Theorem 1. *Suppose that the assumptions* (I), (II) *and* (III) *hold, and that these functions satisfy the following conditions*:

(1) $0<f_0\leqq\frac{f(x)}{x}\quad(x\neq 0),\quad f'(x)\leqq f_1\qquad in\ R^1.$

(2) $0<\phi_0\leqq\frac{\tilde{\phi}(x,y)}{y}\leqq\phi_1\quad(y\neq 0),\quad\tilde{\phi}_x(x,y)\leqq 0\qquad in\ R^2.$

(3) $0<\psi_0\leqq\tilde{\psi}(x,y,z)\leqq\psi_1,\quad\tilde{\psi}_x(x,y,z)y\leqq 0$ *and* $y\tilde{\psi}_z(x,y,z)\geqq 0$ *in* R^3.

(4) $0<c_0\leqq c(t)\leqq c_1,\quad 0<b_0\leqq b(t)\leqq b_1,\quad 0<a_0\leqq a(t)\leqq a_1\qquad in\ I.$

(5) $\sup_{y\neq 0}\frac{1}{y}\{\phi_1(x,y)-\phi_0(x,y)\}=p<+\infty,$
$\sup\{\psi_1(x,y,z)-\psi_0(x,y,z)\}=q<+\infty.$

(6) $a_0b_0\phi_0\psi_0>c_1f_1.$

(7) $\limsup_{t\to\infty}\frac{1}{t}\int_0^t\{|a'(s)|+|b'(s)|+|c'(s)|\}ds$ *has an infinitesimal upper bound.*

(8) $|p(t,x,y,z)|\leqq p_1(t)+p_2(t)\cdot(x^2+y^2+z^2)^{\alpha/2}+\Delta\cdot(x^2+y^2+z^2)^{1/2}$ *where* α *and* Δ *are constants such that* $0\leqq\alpha<1$ *and* $\Delta\geqq 0$, *and* $p_1(t),p_2(t)$ *are non-negative, continuous functions defined in* $I=[0,+\infty)$.

(9) *For a positive number* ω, *and for some* $T>0$,

$$\int_T^t e^{-\omega(t-s)}\{p_1(s)\}^r\,ds\to 0\quad\text{as } t\to\infty\quad(1\leqq r\leqq 2),$$

$$\int_T^t e^{-\omega(t-s)}\{p_2(s)\}^{(r/1-\alpha)}\,ds\to 0\quad\text{as } t\to\infty\quad(1\leqq r\leqq 2),$$

If Δ *is sufficiently small and if*

$$(10)\quad\begin{cases}a_0\psi_0-\mu_1-(2a_1\psi_0+2\mu_1+3a_1)q-2b_1p>0,\\ \mu_1b_0\phi_0-c_1f_1-2\mu_1(a_1q+b_1p)>0,\\ a_0\psi_0(b_0\phi_0-\mu_2)-c_1f_1-a_1\psi_0(a_1q+b_1p)>0,\end{cases}$$

$$\left|\begin{array}{l} c_0 f_0 - \dfrac{\mu_2}{4}\left\{a_1(\psi_1-\psi_0) - \dfrac{b_1}{\mu_1}(\phi_1-\phi_0)\right\} \\ \qquad\qquad\qquad - \dfrac{\mu_2(H2\mu_2)}{4\mu_1}(a_1\mu_1 q + b_1 p) > 0, \end{array}\right.$$

where μ_1 *and* μ_2 *are arbitrarily fixed constants satisfying*

$$a_0\psi_0 > \mu_1 > \frac{c_1 f_1}{b_0\phi_0}, \qquad \frac{a_0 b_0 \phi_0 \psi_0 - c_1 f_1}{a_0\psi_0} > \mu_2 > 0,$$

then every solution of (1.1) *is uniform-bounded and satisfies*

$$\{x(t)\}^2 + \{\dot{x}(t)\}^2 + \{\ddot{x}(t)\}^2 \to 0 \quad \text{as } t\to\infty. \tag{2.1}$$

Theorem 2. *Suppose that the assumptions* (I), (II) *and* (III), *and the conditions* (1)–(7), (10) *of Theorem* 1 *hold, then the zero solution of* (1.2) *is asymptotically stable in the large as* $t\to\infty$, *if* $\phi(t,x,0)=0$ *in* $I\times R^1$.

Remark. The condition (9) is weaker than the diminishing condition:

$$\int_t^{t+1}\{p_1(t)\}^r ds \to 0, \quad \int_t^{t+1}\{p_2(t)\}^{(r/1-\alpha)} ds \to 0 \qquad \text{as } t\to\infty,$$

(see F. Brauer and J. A. Nohel [1]).

3. Proof of Theorems. We require first the following

Lemma 1. *Consider the system of differential equations*

$$\dot{x} = f(t,x), \qquad f(t,x)\in C[I\times R^n]. \tag{3.1}$$

If there exists a Liapunov function $U(t,x)$ *satisfying*

i) $U(t,x)\in C^1[I\times R^n]$,

ii) $a(\|x\|) \leqq U(t,x) \leqq b(\|x\|)$ *where* $a(r), b(r)\in CIP$ (*the family of continuous increasing positive definite functions*) *and* $a(r)\to\infty$ *as* $r\to\infty$,

iii) $\dot{U} \leqq (-\lambda+\lambda_1(t))U + \lambda_2(t)U^{\mu/2}$ $(0\leqq\mu<2)$ *where* $\lambda_i(t)\in C[I]$ *and*

(1) $\displaystyle\limsup_{t\to\infty}\frac{1}{t}\int_0^t \lambda_1(s)ds = \lambda_0 < \lambda$,

(2) $\displaystyle e^{-\omega t}\int_T^t e^{\omega s}\lambda_2(s)ds \to 0$ *as* $t\to\infty$ *for some* $\omega>0$ *and for some* $T>0$,

then, all solutions $x(t)$ *of* (3.1) *are uniform-bounded and satisfy* $x(t)\to 0$ *as* $t\to\infty$.

Proof of Lemma 1. Let

$$V(t,x) = U(t,x)e^{-\varepsilon t}\int_t^\infty e^{\varepsilon s}\exp\left(-\lambda(s-t) + \int_t^s \lambda_1(\tau)d\tau\right)ds.$$

Then for some positive constant K

$$e^{\lambda}\cdot U(t,x) \leqq V(t,x) \leqq e^K\cdot U(t,x) \qquad \text{in } I\times R^n,$$

and

$$\frac{dV(t,x(t))}{dt} \leqq -\varepsilon V(t,x(t)) + \lambda_2(t)K^{\mu/2}\cdot V^{\mu/2}(t,x(t)) \qquad \text{in } I\times R^n$$

where ε is a constant such that for some $T'>T$

$$\lambda-\frac{1}{t-T}\int_T^t \lambda_2(s)ds \geqq 2\varepsilon \qquad \text{for all } t\geqq T'.$$

Thus, let $W=V^{1-(\mu/2)}$, we have for all $t\geqq T'$

$$W(t,T,W_0)\leqq W_0 e^{-(2/2-\mu)\varepsilon(t-T)}+\int_T^t e^{-(2/2-\mu)\varepsilon(t-s)}\lambda_2(s)ds$$

and $W(t,x(t))$ is bounded and tends to zero as $t\to\infty$. Therefore all solutions $x(t)$ of (3.1) are uniform-bounded and satisfy $x(t)\to 0$ as $t\to\infty$. Q.E.D.

Proof of Theorem 1. First we note that the equation (1.1) is equivalent to the following system of differential equations:

(3.2) $\dot{x}=y,\ \dot{y}=z,\ \dot{z}=p(t,x,y,z)-c(t)f(x)-\phi(t,x,y)-\psi(t,x,y,z).$

We consider now the Liapunov function

$$U(t,x,y,z)=U_1(t,x,y,z)+U_2(t,x,y,z)+U_3(t,x,y,z)$$

where

$$\begin{aligned}2U_1(t,x,y,z)=&2\mu_1 c(t)\int_0^x f(\xi)d\xi+2c(t)f(x)y+2b(t)\int_0^y \tilde{\phi}(x,\eta)d\eta\\&+2\mu_1 a(t)\int_0^y \tilde{\psi}(x,\eta,0)\eta d\eta+2\mu_1 yz+z^2,\end{aligned} \tag{3.3}$$

$$\begin{aligned}2U_2(t,x,y,z)=&\mu_2 b(t)\phi_0 x^2+2a(t)c(t)\psi_0\int_0^x f(\xi)d\xi+a^2(t)\psi_0^2 y^2\\&-\mu_2 y^2+2b(t)\int_0^y \tilde{\phi}(x,\eta)d\eta+z^2+2\mu_2 a(t)\psi_0 xy\\&+2\mu_2 xz+2a(t)\psi_0 yz+2c(t)f(x)y,\end{aligned} \tag{3.4}$$

$$2U_3(t,x,y,z)=2a^2(t)\psi_0\int_0^y \tilde{\psi}(x,\eta,0)\eta d\eta-a^2(t)\psi_0^2 y^2. \tag{3.5}$$

The inequality

$$\begin{aligned}2U_1=&2c(t)\int_0^x(\mu_1-\lambda f'(\xi))^2 f(\xi)\,d\xi+2c(t)\left\{\sqrt{\lambda}\,f(x)+\frac{y}{\sqrt{\lambda}}\right\}^2\\&+(\mu_1 y+z)^2+\frac{2}{\lambda}\int_0^y(\lambda b(t)\tilde{\phi}(x,\eta)-c(t)\eta)d\eta\\&+2\mu_1\int_0^y\{a(t)\tilde{\psi}(x,\eta,0)-\mu_1\}\eta d\eta\\ \geqq& c_0(\mu_1-\lambda f_1)f_0 x^2+\sigma z^2+\left\{\frac{1}{\lambda}(\lambda b_0\phi_0-c_1)+\mu_1\left(a_0\psi_0-\frac{\mu_1}{1-\sigma}\right)\right\}y^2\end{aligned}$$

implies that there exists a constant δ_0 such that

$$\delta_0\cdot(x^2+y^2+z^2)\leqq U_1(t,x,y,z), \quad \text{if} \quad a_0\psi_0(1-\sigma)>\mu_1 \quad \text{and} \quad \mu_1>\lambda f_1>\frac{c_1 f_1}{b_0\phi_0}.$$

The existence of δ_1 such that $U_1(t,x,y,z)\leqq\delta_1\cdot(x^2+y^2+z^2)$ is obvious. Next we have

$$2U_2\geqq\mu_2(b_0\phi_0-\mu_2)x^2+\frac{a_0b_0\phi_0\psi_0-a_0\psi_0\mu_2-c_1f_1}{a_0\psi_0}\cdot y^2,$$

and by the conditions there exist constants δ_2,δ_3 satisfying

$$\delta_2\cdot(x^2+y^2)\leqq U_2(t,x,y,z)\leqq\delta_3(x^2+y^2+z^2).$$

By the same argument we have for some $\delta_4>0$,

$$0\leqq U_3(t,x,y,z)\leqq\delta_4\cdot(x^2+y^2+z^2).$$

Thus we have

$$\delta_0\cdot(x^2+y^2+z^2)\leqq U(t,x,y,z)\leqq\delta_5\cdot(x^2+y^2+z^2) \tag{3.6}$$

for some positive constants δ_0 and δ_5.

Next along the solution of (3.2),

$$\begin{aligned}\dot U_{(3.2)}\leqq &-[\mu_2c(t)xf(x)+\{\psi(t,x,y,z)-\mu_1z\}z\\ &+\{\psi(t,x,y,z)-a(t)\psi_0z\}z+\{\mu_1\phi(t,x,y)-c(t)f'(x)y\}y\\ &+\{a(t)\psi_0\phi(t,x,y)-c(t)f'(x)y-\mu_2a(t)\psi_0y\}y\\ &+\mu_2\{\phi(t,x,y)-b(t)\phi_0y\}x+\mu_1\{\psi(t,x,y,z)-a(t)\tilde\psi(x,y,0)z\}y\\ &+\mu_2\{\psi(t,x,y,z)-a(t)\psi_0z\}x+a(t)\psi_0\{\psi(t,x,y,z)-a(t)\psi_0z\}y\\ &+2\{\phi(t,x,y)-b(t)\tilde\phi(x,y)\}z+a^2(t)\psi_0\{\tilde\psi(x,y,0)-\psi_0\}yz]\\ &+D_1\cdot\{(x^2+y^2+z^2)^{1/2}\,|p(t,x,y,z)|\\ &+D_2\cdot\{|a'(t)|+|b'(t)|+|c'(t)|\}(x^2+y^2+z^2)\end{aligned}$$

for some positive constants D_1 and D_2.

Therefore by the inequality (3.6) and condition (10), we obtain the following estimate for some positive constant $\delta_6,\delta_7,\delta_8$:

$$\dot U_{(3.2)}=-[\delta_6-\delta_7\{|a'(t)|+|b'(t)|+|c'(t)|\}]U+\delta_8\,|p(t,x,y,z)|\cdot U^{1/2}.$$

Thus we have

$$\begin{aligned}\dot U_{(3.2)}\leqq &-\{\delta_6-\delta_7(|a'(t)|+|b'(t)|+|c'(t)|)\}U+\delta_8\cdot p_1(t)U^{1/2}\\ &+\delta_9p_2(t)U^{(1+\alpha)/2}+\delta_{10}\cdot\varDelta\cdot U\\ \leqq&\{-\delta_6+\delta_{10}\cdot\varDelta+\varepsilon+\delta_7(|a'(t)|+|b'(t)|+|c'(t)|)\}U\\ &+\delta_8\left(p_1(t)-\frac{\varepsilon}{2\delta_8}U^{1/2}\right)U^{1/2}+\delta_9\left\{p_2(t)-\frac{\varepsilon}{2\delta_9}U^{(1+\alpha)/2}\right\}U^{(1+\alpha)/2}\end{aligned}$$

and

$$\begin{aligned}\dot U_{(3.2)}\leqq&\{-\delta_6+\delta_{10}\cdot\varDelta+\varepsilon+\delta_7(|a'(t)|+|b'(t)|+|c'(t)|)\}U\\ &+\delta_{11}\cdot U^{\mu/2}\{p_1(t)\}^{2-\mu}+\delta_{12}U^{\mu/2}\{p_2(t)\}^{(2-\mu)/(1-\alpha)}\end{aligned} \tag{3.7}$$

where $0\leqq\mu\leqq1, 0\leqq\alpha<1, \varepsilon>0$ such that $\delta_6-\delta_{10}\cdot\varDelta-\delta_7\cdot\lambda_0-2\varepsilon>0$, where

$$\lambda_0=\limsup_{t\to\infty}\frac{1}{t}\int_0^t(|a'(s)|+|b'(s)|+|c'(s)|)ds.$$

The Lemma 1 will be used to complete the proof of Theorem 1. Q.E.D. Theorem 2 is an immediate consequence of Theorem 1.

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