88. *An Example of Temporally Inhomogeneous Scattering*

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§1. The result. Consider a system of linear partial differential equations

$$\frac{\partial u(x,t)}{\partial t}=\sum_{j=1}^{n} A_j(x,t)\frac{\partial u(x,t)}{\partial x_j}+B(x,t)u(x,t). \tag{1.1}$$

Here $u=(u_1,\cdots,u_N)$ is an N-vector of unknown functions of x and t; $A_j(x,t)$ and $B(x,t)$ are $N\times N$ matrix functions, and $A_j(x,t)$ are assumed to be Hermitian symmetric.

In order to guarantee the existence and the uniqueness of the solution $u(x,t)\in\mathcal{E}_t^1(L^2(\boldsymbol{R}^n))\cap\mathcal{E}_t^0(H^1(\boldsymbol{R}^n))$[1)] of (1.1) with Cauchy data $u(x,0)=u_0(x)\in H^1(\boldsymbol{R}^n)$, we assume the following (see [5], [6]):

(I) (a) The maps $t\mapsto A_j(\cdot,t)$ are continuous on $(-\infty,\infty)$ to $\mathcal{B}^1(\boldsymbol{R}^n)$,

(b) $t\to B(\cdot,t)$ is continuous on $(-\infty,\infty)$ to $\mathcal{B}^0(\boldsymbol{R}^n)$ and

$$\frac{\partial B(x,t)}{\partial x_j}\in\mathcal{B}^0(\boldsymbol{R}^n\times(-\infty,\infty)),\qquad j=1,2,\cdots,n.$$

Here $\mathcal{B}^l(\boldsymbol{R}^m)$ is the set of all $N\times N$-matrix valued functions A such that A and $D^\alpha A$, $|\alpha|\leqq l$ are continuous and bounded on $\boldsymbol{R}^m$.

We further consider two systems of linear partial differential equations given by

$$\frac{\partial u^\pm(x,t)}{\partial t}=\sum_{j=1}^{n} A_j^\pm\frac{\partial u^\pm(x,t)}{\partial x_j}+B^\pm u^\pm(x,t) \tag{1.2$^\pm$}$$

where $A_j^\pm$ are $N\times N$ constant Hermitian symmetric matrices and $B^\pm$ are $N\times N$ constant matrices satisfying $B^\pm+(B^\pm)^*=0$. (F^* denotes the Hermitian conjugate matrix of F.)

We assume that (1.2)$^\pm$ are close to (1.1) near $|t|=\infty$ in the following sense.

(II) There exists a function $\phi(t)\in L^1(-\infty,\infty)$ satisfying

(1.3) $|A_j(x,t)-A_j^\pm|_{\mathcal{B}^1(\boldsymbol{R}^n)}\leqq\phi(t)$, $|B(x,t)-B^\pm|_{\mathcal{B}^1(\boldsymbol{R}^n)}\leqq\phi(t)$ for $t\gtrless 0$.

We define an operator $U(t;s)$ by $U(t;s)u_0=u(x,t)$ where $u(x,t)\in\mathcal{E}_t^1(L^2(\boldsymbol{R}^n))\cap\mathcal{E}_t^0(H^1(\boldsymbol{R}^n))$ is a solution of (1.1) with Cauchy data $u_0(x)\in H^1(\boldsymbol{R}^n)$ at time s. We define the operators $U_0^\pm(t;s)$ analogously. By the energy inequality, expressed in Lemma 1 and Lemma 2 below, the

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1) $u(x,t)\in\mathcal{E}_t^l(H^k(\boldsymbol{R}^n))$ means that $u(\cdot,t)$ is a $H^k(\boldsymbol{R}^n)$ valued function of t, l-times continuously differentiable with respect to t in $H^k(\boldsymbol{R}^n)$-norm.

operators $U(t;s)$ and $U_0^{\pm}(t;s)$ are well defined and are extended as the bounded operators in $L^2(\boldsymbol{R}^n)$. Our Theorem 1 reads as follows:

Theorem 1. *We assume* (I) *and* (II). *Then, there exist operators* $W_{\pm}$, ${}^*W_{\pm}$ *defined as follows*:

$$ (1.4)\qquad {}^*W_+=\lim_{t\to\infty} {}^*W_+(t), \qquad {}^*W_+(t)=U_0^+(0;t)U(t;0), $$

$$ (1.5)\qquad W_-=\lim_{t\to-\infty} W_-(t), \qquad W_-(t)=U(0;t)U_0^-(t;0), $$

$$ (1.6)\qquad W_+=\lim_{t\to\infty} W_+(t), \qquad W_+(t)=U(0;t)U_0^+(t;0), $$

$$ (1.7)\qquad {}^*W_-=\lim_{t\to-\infty} {}^*W_-(t), \qquad {}^*W_-(t)=U_0^-(0;t)U(t;0), $$

limits being taken in $L^2(\boldsymbol{R}^n)$.

In order to state our Theorem 2, we introduce another notion of solutions.

Definition. A function $\tilde{u}^{\pm}(x,t)\in\mathcal{E}_t^0(L^2(\boldsymbol{R}^n))$ is said to be a weak solution of $(1.2)^{\pm}$ if it satisfies

$$ (1.8)^{\pm}\qquad \begin{aligned} &\int_{\boldsymbol{R}^n}\tilde{u}^{\pm}(x,t)\overline{\varphi^{\pm}(x,t)}dx-\int_{\boldsymbol{R}^n}\tilde{u}^{\pm}(x,s)\overline{\varphi^{\pm}(x,s)}dx\\ &\quad=\int_s^t d\tau\int_{\boldsymbol{R}^n}\tilde{u}^{\pm}(x,\tau)\overline{\left[\frac{\partial\varphi^{\pm}(x,\tau)}{\partial\tau}-\sum_{j=1}^n A_j^{\pm}\frac{\partial\varphi^{\pm}(x,\tau)}{\partial x_j}-B^{\pm}\varphi^{\pm}(x,\tau)\right]}dx \end{aligned} $$

for any $s,t\in(-\infty,\infty)$ and $\varphi^{\pm}(x,t)\in\mathcal{E}_t^1(L^2(\boldsymbol{R}^n))\cap\mathcal{E}_t^0(H^1(\boldsymbol{R}^n))$.

Then, we have

Theorem 2. *We assume* (I) *and* (II). *Let* $u(x,t)\in\mathcal{E}_t^1(L^2(\boldsymbol{R}^n))\cap\mathcal{E}_t^0(H^1(\boldsymbol{R}^n))$ *be a solution of* (1.1). *If there exists a function* $u^-(x,t)\in\mathcal{E}_t^0(L^2(\boldsymbol{R}^n))$ *which is a weak solution of* $(1.2)^-$ *satisfying*

$$ (1.9)\qquad \lim_{t\to-\infty}\|u(x,t)-u^-(x,t)\|_{L^2(\boldsymbol{R}^n)}=0, $$

then there exists a uniquely defined function $u^+(x,t)\in\mathcal{E}_t^0(L^2(\boldsymbol{R}^n))$ *which is a weak solution of* $(1.2)^+$ *satisfying*

$$ (1.10)\qquad \lim_{t\to\infty}\|u(x,t)-u^+(x,t)\|_{L^2(\boldsymbol{R}^n)}=0. $$

§2. The sketch of the proofs. We prepare the following two lemmas.

Lemma 1. *Let* $u(x,t)\in\mathcal{E}_t^1(L^2(\boldsymbol{R}^n))\cap\mathcal{E}_t^0(H^1(\boldsymbol{R}^n))$ *be a solution of* (1.1). *Then we have,*

$$ (2.1)\qquad \|u(x,t)\|\leqq\exp\left(\int_0^t\phi(s)ds\right)\|u(x,0)\| $$

$$ (2.2)\qquad \begin{aligned} &\|u(x,t)\|+\sum_{k=1}^n\|u_k(x,t)\|\\ &\quad\leqq\exp\left((2+n)\int_0^t\phi(s)ds\right)\cdot\left[\|u(x,0)\|+\sum_{k=1}^n\|u_k(x,0)\|\right] \end{aligned} $$

where $u_k(x,t)=\dfrac{\partial u(x,t)}{\partial x_k}$, $\|v\|^2=(v,v)=\int_{\boldsymbol{R}^n}v(x)\overline{v(x)}dx$

Proof. We have

$$\left|\frac{1}{2}\frac{d}{dt}\|u(\cdot,t)\|^2\right|=\left|\operatorname{Re}\left(\sum_{j=1}^{n}(A_j(x,t)-A_j^{\pm})u_j,u\right)\right.$$
$$(2.3)\qquad \left.+\operatorname{Re}((B(x,t)-B^{\pm})u,u)+\operatorname{Re}\left(\sum_{j=1}^{n}A_j^{\pm}u_j+B^{\pm}u,u\right)\right|$$
$$\leqq\phi(t)\cdot\|u(\cdot,t)\|^2.$$

Thus, since the inequality $\gamma'(t)\leqq\phi(t)\gamma(t)$ for non-negative integrable functions implies $\gamma(t)\leqq\gamma(0)\exp\left(\int_0^t\phi(s)ds\right)$, we obtain (2.1).

Assume further that $u(x,t)\in\mathcal{E}_t^1(H^1(\boldsymbol{R}^n))\cap\mathcal{E}_t^0(H^2(\boldsymbol{R}^n))$. Then, we have

$$\left|\frac{1}{2}\frac{d}{dt}\|u_k(\cdot,t)\|^2\right|=\left|\operatorname{Re}\left(\sum_{i=1}^{n}(A_j(x,t)-A_j^{\pm})u_{kj},u_k\right)\right.$$
$$+\operatorname{Re}((B(x,t)-B^{\pm})u_k,u_k)$$
$$(2.4)\qquad +\operatorname{Re}\left(\sum_{j=1}^{n}A_j^{\pm}u_{kj}+B^{\pm}u_k,u_k\right)$$
$$\left.+\operatorname{Re}\left(\sum_{j=1}^{n}A_j^{(k)}(x,t)u_j+B^{(k)}(x,t)u,u_k\right)\right|.$$

Combining this with (2.3) and (1.3), we have

$$\frac{d}{dt}\left[\|u(\cdot,t)\|+\sum_{k=1}^{n}\|u_k(\cdot,t)\|\right]$$
$$(2.5)\qquad \leqq(n+2)\cdot\phi(t)\left[\|u(\cdot,t)\|+\sum_{k=1}^{n}\|u_k(\cdot,t)\|\right].$$

As in the case of (2.1), we obtain (2.2) from (2.5). Using the Friedrich mollifier with respect to x, we can remove the additional regularity for $u(x,t)$. Q.E.D.

Lemma 2 ([1], [3]). *Let $u^{\pm}(x,t)$ be a weak solution of* (1.2)$^{\pm}$. *Then we have*

$$(2.6)\qquad \|u^{\pm}(x,t)\|=\|u^{\pm}(x,s)\|\qquad \textit{for any } s,t.$$

Moreover, if $u^{\pm}(x,t)\in\mathcal{E}_t^1(L^2(\boldsymbol{R}^n))\cap\mathcal{E}_t^0(H^1(\boldsymbol{R}^n))$, we have

$$(2.7)\quad \|u^{\pm}(x,t)\|_1=\|u^{\pm}(x,s)\|_1\qquad \textit{for any } s,t, \textit{ where } \|\cdot\|_1=\|\cdot\|_{H^1(\boldsymbol{R}^n)}.$$

Proof. Let $\{u_m^{\pm}(x)\}$ be a sequence in $H^1(\boldsymbol{R}^n)$ converging to $u^{\pm}(x,s)$ in $L^2(\boldsymbol{R}^n)$-norm. Define $\varphi_m^{\pm}(x,t)=U_0^{\pm}(t,s)u_m^{\pm}(x)$. Putting $\varphi_m^{\pm}(x,t)$ in (1.8)$^{\pm}$ in place of $\varphi^{\pm}(x,t)$, we obtain

$$\int_{\boldsymbol{R}^n}u^{\pm}(x,t)\overline{\varphi_m^{\pm}(x,t)}dx=\int_{\boldsymbol{R}^n}u^{\pm}(x,s)\overline{u_m^{\pm}(x)}dx.$$

Tending m to ∞, and using the Schwarz inequality, we have

$$\|u^{\pm}(x,s)\|\leqq\|u^{\pm}(x,t)\|.$$

As t and s are taken arbitrary, we have (2.6). Similarly as in the proof of (2.2), we obtain (2.7) immediately. Q.E.D.

Proof of Theorem 1. For $u_0(x)\in H^1(\boldsymbol{R}^n)$, we can differentiate $^*W_+(t)u_0$ given by (1.4) and we obtain

$$^*W_+(t)u_0-u_0$$
$$=\int_0^t U_0^+(0;s)\left[\sum_{j=1}^{n}(A_j(x,s)-A_j^+)\frac{\partial}{\partial x_j}+(B(x,s)-B^+)\right]U(s;0)u_0ds$$

Since $u_0 \in H^1(\boldsymbol{R}^n)$, we have (i) $U(t;0)u_0 \in H^1(\boldsymbol{R}^n)$, (ii) $[\cdots]U(t;0)u_0$ is continuous in t and (iii) $\|[\cdots]U(t;0)u_0\|$ is integrable on $(0,\infty)$ by (1.3), (2.2) and (2.6). Therefore applying Theorem X.3.7 of T. Kato [4], we can prove that there exists ${}^*W_+ = {}^*W_+(\infty)$.

Analogously, we prove the existence of the operators W_+, W_- and ${}^*W_-$.

Poof of Theorem 2. By Lemma 2, we obtain the following representation for the weak solution $u^{\pm}(x,t)$ of $(1.2)^{\pm}$

$$u^{\pm}(x,t) = U_0^{\pm}(t;s)u^{\pm}(x,s). \tag{2.8}$$

We also have

$$u(x,t) = U(t;s)u(x,s). \tag{2.9}$$

Therefore, by (1.7), (1.9) and (2.1), we obtain $u(x,0) = W_- u^-(x,0)$, because we have

$$\begin{aligned} &\|U(t;0)u(x,0) - U_0^-(t;0)u^-(x,0)\| \\ &\quad = \|U(t;0)[u(x,0) - U(0;t)U_0^-(t;0)u^-(x,0)]\| \\ &\quad \geqq \exp\left(-\int_0^t \phi(s)ds\right)\|u(x,0) - U(0;t)U_0^-(t;0)u^-(x,0)\|. \end{aligned}$$

Thus, by defining $u^+(x,t) = U_0^+(t;0)({}^*W_+)W_- u^-(x,0)$, we prove immediately that $u^+(x,t)$ satisfies $(1.2)^+$ and (1.1). Q.E.D.

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