110. On Nonexistence of Global Solutions of Some Semilinear Parabolic Differential Equations

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The purpose of this paper is to show that *the semilinear parabolic equation* $(\partial/\partial t)u = \Delta u + u^{1+\alpha}$ *has no global solutions for any nontrivial nonnegative initial data* $u_0(x)$ *in case of* $N=2$, $\alpha=1$ *or* $N=1$, $\alpha=2$, *where* N *denotes the dimension of x-space.*
This problem was considered in Fujita H. [1] and in more general form [2]. The conclusions of [1] are as follows.

In case of $N\alpha<2$ there does not exist a global solution for any nontrivial nonnegative initial data. On the other hand, in case of $N\alpha>2$, there exists a global solution for sufficiently small initial data, and no global solutions for sufficiently large initial data.

This paper will give a partial settlement for the case $N\alpha=2$.

We consider the next problem.

$$(1)\qquad \frac{\partial}{\partial t}u(t,x) = \Delta u(t,x) + u(t,x)^{1+\alpha} \qquad (t,x)\in[0,T)\times R^N,$$
$$u(0,x)=u_0(x),$$

where $u_0(x)$ is a nonnegative bounded continuous function.
A function $u=u(t,x)$ is said to be a solution of (1) if the following (i) and (ii) hold (see [1] or [2]);

(i) u is bounded and continuous in $[0,T']\times R^N$, where T' is an arbitrary constant $<T$. The initial condition is satisfied in the usual sense.

(ii) The differential equation is satisfied by u in the distribution sense in $(0,T)\times R^N$.
The "global solution" means the solution of (1) for $T=\infty$.

Theorem. *In case of* $N=2$, $\alpha=1$ *or* $N=1$, $\alpha=2$, *the initial value problem* (1) *has no global solutions for any nontrivial initial data* u_0.

The remainder of this paper will be devoted to the proof.

The problem (1) is equivalent to the following problem of the integral equation.

$$(2)\qquad \begin{aligned} u(t,x) &= (4\pi t)^{-N/2}\int_{R^N}\exp\left(-|x-y|^2/4t\right)u_0(y)dy \\ &\quad + \int_0^t (4\pi(t-\tau))^{-N/2}\int_{R^N}\exp\left(-|x-y|^2/4(t-\tau)\right) \\ &\quad \times (u(\tau,y))^{1+\alpha}dyd\tau. \end{aligned}$$

For sufficiently small T, (2) has a solution $u(t,x)$. If $u_0 \not\equiv 0$, we can find $t_0>0$, $C_0>0$ and $\beta_0>0$, so that they satisfy $u(t_0,x) \geqq C_0 \exp(-\beta_0|x|^2)$. By the comparison theorem, it is enough to show that (2) has not global solutions with the initial data $u_0(x)=C_0\exp(-\beta_0|x|^2)$. We are going to prove this by using the iterrated estimate from below in the case $N=2$.

Lemma. *If we define* $\{a_k(t)\}_{k=1}^{\infty}$ *by*

$$a_1(t)=C_0/(1+4\beta_0 t),$$

$$(3)\qquad a_{k+1}(t)=\frac{1}{k+1}\,\frac{1}{1+4\beta_0 t}\int_0^t \sum_{r=1}^{k} a_r(\tau)a_{k+1-r}(\tau)(1+4\beta_0\tau)d\tau, \qquad k=1,2,\cdots,$$

we have

$$(4)\qquad u(t,x) \geqq \sum_{k=1}^{n} a_k(t)E(k;t,x) \qquad \textit{for } n=1,2,\cdots,$$

and

$$(5)\qquad a_k(t) \geqq k\, 6^{-k+1}(1+4\beta_0 t)^{-1}4\beta_0(C_0/4\beta_0)^k(\log(1+4\beta_0 t))^{k-1},$$

where $E(k;t,x)=\exp(-k\beta_0|x|^2/(1+4\beta_0 t))=E(1;t,x)^k$.

Proof of lemma. From (2) we have

$$u(t,x) \geqq (4\pi t)^{-1}\int_{R^2}\exp(-|x-y|^2/4t)\,C_0\exp(-\beta_0|y|^2)dy$$
$$=\frac{C_0}{1+4\beta_0 t}\exp(-\beta_0|x|^2/(1+4\beta_0 t))=a_1(t)E(1;t,x).$$

Now we assume that (4) is true for some n, then by (2) we have

$$u(t,x) \geqq a_1(t)E(1;t,x)$$
$$+\int_0^t (4\pi(t-\tau))^{-1}\int_{R^2}\exp(-|x-y|^2/4(t-\tau))$$
$$\times\left(\sum_{k=1}^{n} a_k(\tau)E(k;\tau,y)\right)^2 dy d\tau.$$

Then noting that

$$E(r;t,x)E(k+1-r;t,x)=E(k+1;t,x)$$

and

$$(4\pi(t-\tau))^{-1}\int_{R^2}\exp(-|x-y|^2/4(t-\tau))\,E(k+1,\tau,y)dy$$
$$=\frac{1+4\beta_0\tau}{1+4(k+1)\beta_0 t-4k\beta_0\tau}\exp(-(k+1)\beta_0 x^2/(1+4(k+1)\beta_0 t-4k\beta_0\tau))$$
$$\geqq (1+4\beta_0\tau)(k+1)^{-1}(1+4\beta_0 t)^{-1}E(k+1;t,x) \qquad \text{for } 0 \leqq \tau < t,$$

we get

$$u(t,x) \geqq a_1(t)E(1;t,x)$$
$$+\sum_{k=1}^{n}\frac{1}{(k+1)(1+4\beta_0 t)}\int_0^t\sum_{r=1}^{k} a_r(\tau)a_{k+1-r}(\tau)(1+4\beta_0\tau)d\tau E(k+1;t,x)$$
$$=\sum_{k=1}^{n+1} a_k(t)E(k;t,x).$$

(5) can be proved from (3) by induction.

From (4) and (5) we have the following estimate for $|x|\leqq M$.

$$u(t, x)\geqq C_1 \sum_{k=1}^{n} k(C_0 \exp(-\beta_0 M^2)/24\beta_0)^{k-1} (\log(1+4\beta_0 t))^{k-1}.$$

Since the radius of convergence of the series $\sum_{k=1}^{\infty} k(C_0 \exp(-\beta_0 M^2)/24\beta_0)^{k-1} z^{k-1}$ is equal to $\rho=24\beta_0 \exp(\beta_0 M^2)/C_0$, the right-hand-side of the previous estimate tends to ∞ as n goes to ∞ for $t>(e^{\rho}-1)/4\beta_0$. This proves the nonexistence of the global solution of (1) in the case $N=2$.

Similarly, in case of $N=1$, $\alpha=2$, we can get the following estimates for any $n=1, 2, \cdots$.

$$u(t, x)\geqq \sum_{k=0}^{n} A_k(t) E(2k-1; t, x). \tag{6}$$

Here, coefficients $A_k(t)$ satisfy the followings.

$$\begin{aligned} &A_0(t)=C_0/\sqrt{1+4\beta_0 t}, \\ &A_{k+1}(t)=\frac{1}{2k+3}\,\frac{1}{1+4\beta_0 t}\int_0^t \sum_{j+m+n=k} A_j(\tau)A_m(\tau)A_n(\tau)\sqrt{1+4\beta_0\tau}\,d\tau. \end{aligned} \tag{7}$$

From this we have

$$u(t, x)\geqq \sum_{k=0}^{n} (k+1)K^k \frac{L}{1+4\beta_0 t} (\log(1+4\beta_0 t))^k E(1: t, x)^{2k+1},$$

where K and L are positive constants depending only on C_0 and β_0. This also shows the nonexistence of the global solution of (1) in the case $N=1$, $\alpha=2$.

References

[1] Fujita, H.: On the blowing up of solutions of the Cauchy problem for $u_t=\Delta u+u^{1+\alpha}$. J. Fac. Sci. Univ. Tokyo Sect. I, **13**, 109–124 (1966).

[2] ——: On some nonexistence and nonuniqueness theorems for nonlinear parabolic equations. Proc. Symp. Pure Math. A. M. S., **18**, 105–113.