105. *On Prehomogeneous Compact Kähler Manifolds*

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1. In this note we establish some results on classification of compact complex prehomogeneous Kähler manifolds. Details will appear elsewhere. By a compact prehomogeneous manifold, we mean a compact complex manifold whose automorphism group has an open orbit. In [4], J. Potters classified prehomogeneous compact complex surfaces. In what follows we shall state a couple of structure theorems on prehomogeneous compact Kähler manifolds and a classification of such manifolds with coirregularity less than 3.

For convenience sake, we list here some notations and terminologies used below. Let V be a compact complex manifold.

$\mathrm{Aut}^\circ(V)=$ the connected biholomorphic automorphism group of V.

$A(V)=$ the Albanese torus of V.

$q(V)=\dim H^1(V,\mathcal{O})$ which is called the irregularity of V.

$cq(V)=\dim V-q(V)$ which is called the coirregularity of V.

By a regular manifold we mean a compact complex manifold whose irregularity vanishes. For a complex analytic vector bundle $\boldsymbol{E}$ on V, we denote by $\boldsymbol{P}(\boldsymbol{E})$ the projective bundle associated with $\boldsymbol{E}$.

2. First we state certain general theorems on prehomogeneous manifolds. The following Proposition 1 can be proved by using a lemma due to R. Remmert and van de Ven (see, Potters [4]).

Proposition 1. *A compact complex prehomogeneous manifold is a locally trivial analytic fibre bundle over a compact complex solvmanifold whose fibre is prehomogeneous with trivial Albanese torus.*

Corollary. *A compact Kähler prehomogeneous manifold V is a holomorphic fibre bundle over its Albanese torus $A(V)$ with a regular prehomogeneous fibre.*

In fact every Kähler solvmanifold is isomorphic to a complex torus.

In what follows we always assume that V is Kähler.

Proposition 2. *If $q(V)=0$, then V is a unirational projective variety.*

Proof. For the projectivity of V, see Oeljekraus [3]. We prove the unirationality. Since V is regular, V can be imbedded into a projective space $\boldsymbol{P}^n$ such that this imbedding induces an inclusion of $G=\mathrm{Aut}^\circ(V)$ into $PGL(n)$. This shows that G and its stabilizer subgroup at every point of V are both linear algebraic groups. Since by assumption V is birationally equivalent to an algebraic quotient space of G, V is unirational by a theorem of Chevalley.

Corollary. *Furthermore if* $\dim V \leqq 3$, *then* V *is rational.*

In fact, since $\mathrm{Aut}^\circ(V)$ contains a connected linear algebraic group of positive dimension, V is ruled. Hence, by Proposition 2, V is rational.

Theorem 1. *Let V be a compact Kähler prehomogeneous manifold. Then the Albanese fibration of V is locally flat.*

Outline of the proof. We need the following lemma.

Lemma. *Under the assumption of Theorem* 1, *V can be imbedded into a bundle-homogeneous projective bundle W over $A(V)$ such that this imbedding induces an inclusion of* $\mathrm{Aut}^\circ(V)$ *into* $\mathrm{Aut}^\circ(W)$.

From the above lemma we can derive that kernel B of the canonical epimorphism from $\mathrm{Aut}^\circ(V)$ onto $\mathrm{Aut}^\circ(A(V))$ has only a finite number of connected components. Then the radical R of the maximal compact subgroup of $\mathrm{Aut}^\circ(V)$ is mapped onto $\mathrm{Aut}^\circ(A(V))$. Hence we can reduce the structure group of the Albanese fibring to the complexification of R, which is abelian. This proves Theorem 1 in view of Murakami's theorem [2]. q.e.d.

3. Now we shall give a classification of compact Kähler prehomogeneous manifolds V with $cq(V) \leqq 2$.

Theorem 2. *If* $cq(V)=0$, *then V is a complex torus.*

This is an immediate consequence of the corollary to Proposition 1.

Now assume that $cq(V)=1$ or 2. Note that then, by Proposition 2 and the corollary to Proposition 1, the Albanese fibre is $\boldsymbol{P}^1$ or a rational surface. Moreover in the case where $cq(V)=2$, we can prove that, by blowing down the exceptional submanifolds, the Albanese fibre can be assumed to be relatively minimal.

Theorem 3. *Let V be a compact Kähler prehomogeneous manifold. Suppose that* $cq(V)=1$ *or* 2. *Moreover in the case where* $cq(V)=2$, *we assume that the Albanese fibre is* $\boldsymbol{P}^2$. *Then there exists a flat vector bundle E of rank* 2 *or* 3 *(according to the dimension of the Albanese fibre) on $A(V)$ such that V is isomorphic to* $\boldsymbol{P}(E)$. *Conversely every projective bundle on a complex torus associated with a flat vector bundle is prehomogeneous.*

Outline of the proof. In either case, we can show that there exists a complex torus T in V which is mapped onto $A(V)$ by the Albanese map α and is stable under $G=\mathrm{Aut}^\circ(V)$. Suppose first that $\alpha|_T : T \to A(V)$ is an isomorphism. Then T is a G-stable cross-section of the Albanese fibration. Hence there exist a vector bundle E of rank 2 or 3 on $A(V)$ and an exact sequence of vector bundles

$$0 \to \boldsymbol{1} \to \boldsymbol{E} \to \boldsymbol{F} \to 0$$

where "**1**" denotes the trivial sub-line bundle corresponding to T and $\boldsymbol{F}$ is isomorphic to the normal bundle of T in V. Since T is G-stable, $\boldsymbol{F}$ is homogeneous and hence is flat (Morimoto [1]). From this we can derive that $\boldsymbol{E}$ is also flat by computing the above extension. Next suppose that $\alpha|_T$ is not bijective. In this case we can derive a contradiction by computing the dimension of $\mathrm{Aut}^\circ(V)$ explicitly. Thus Theorem 3 is proved.

Similarly we can determine compact prehomogeneous Kähler manifolds whose Albanese fibres are Hirzebruch manifolds. The classification of such manifolds is given by writing out the representatives of transition functions of fibre bundles explicitly. Furthermore, using the explicit forms, we can prove the following fact which is expected to be true in higher dimensional cases.

Theorem 4. *Every compact Kähler prehomogeneous manifold V with $cq(V)=2$ is obtained from a prehomogeneous $\boldsymbol{P}^2$-bundle over $A(V)$ by a finite number of "equivariant" blowing-ups and blowing-downs where by an "equivariant" blowing-up, we mean a blowing-up with a non-singular centre in which the prehomogeneity is not violated.*

References

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