156. *On the Elementary Partitions of the State Set in a Multiple-Input Semiautomaton*

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1. Introduction. Determination of all homomorphic images of a given semiautomaton is equivalent to constructing all admissible partitions of its state set.

For the case of a one-input semiautomaton, there exists an efficient method for the construction of all admissible partitions. This can be done easily by determining all elementary partitions [1], [2].

For the case of a multiple-input semiautomaton, it seems complicated at first sight. But, even in this case, if all elementary partitions can be constructed, we can use the same procedure as the one-input case and we can obtain all admissible partitions.

In this note, we shall give an algorithm for constructing all elementary partitions of the state set in a multiple-input semiautomaton by using known elementary partitions for the one-input case. We shall borrow many notations and terms from [1].

2. Preliminaries. Consider a semiautomaton $A=(S, \Sigma, M)$, where S is a set of states, $\Sigma=\{\sigma_1, \sigma_2, \cdots, \sigma_n\}$ $(n\geq 2)$ is a set of inputs, and M is a set of transition mappings.

Definition 1. Let π be a partition of S. $\tilde{\pi}$ is called the admissible closure of π in A if and only if $\tilde{\pi}=\Pi_{i\in\Lambda}\xi_i$, where $\{\xi_i\,;\, i\in\Lambda\}$ is the set of all admissible partitions in A such that $\pi\leq\xi_i (i\in\Lambda)$.

In section 4, we shall give a method for constructing the admissible closure $\tilde{\pi}$ of π.

Definition 2. An admissible partition $\pi\neq 0$ of S in A, where 0 means the identity partition, is called elementary if and only if for every admissible partition π' of S in A, $0\leq\pi'\leq\pi$ implies $\pi'=0$ or $\pi'=\pi$.

3. Structure of elementary partitions. For the semiautomaton given in the preceding section, we shall construct following one-input semiautomata:

Put $\Sigma_i=\{\sigma_i\}$ and $M_i=\{\sigma_i^A\}=\{\sigma_i^{A_i}\}$ for each natural number i $(1\leq i\leq n)$. Thus, we obtain the one-input semiautomata $A_i=(S, \Sigma_i, M_i)$ $(1\leq i\leq n)$.

For each semiautomaton A_i $(1\leq i\leq n)$, the set of all elementary partitions of S in A_i can be determined by the procedure introduced in [1], [2]. We denote this set by $\mathcal{P}_i$.

We can now prove the following theorem on the structure of an elementary partition of S in A:

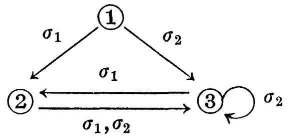
Theorem 1. *If π is an elementary partition of S in A, then there exist elementary partitions $\rho_i \in \mathcal{P}_i$ $(1 \leq i \leq n)$ such that $\tilde{\rho}=\pi$, where $\rho=\sum_{i=1}^{n} \rho_i$.*

Proof. For each natural number i $(1 \leq i \leq n)$, we can consider π as an admissible partition of S in A_i. Thus, there exists an elementary partition $\rho_i \in \mathcal{P}_i$ such that $0 < \rho_i \leq \pi$.

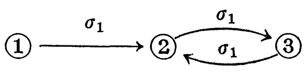
Now, we can take the sum of partitions $\rho=\sum_{i=1}^{n} \rho_i$. Then, it is easy to see that $0 < \rho \leq \pi$. We consider the admissible closure $\tilde{\rho}$ of ρ in A. By virtue of $0 < \rho \leq \pi$ and the admissibility of π in A, we get $0 < \tilde{\rho} \leq \pi$. Since π is elementary in A, $\tilde{\rho}$ must be equal to π. Q.E.D.

Remark. *The converse of the above theorem is not true.* Indeed, there exist some elementary partitions $\rho_i \in \mathcal{P}_i$ $(1 \leq i \leq n)$ such that $\tilde{\rho}$ is not elementary in A, where $\rho=\sum_{i=1}^{n} \rho_i$.

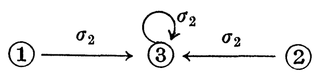
Example. Let $A=(\{1, 2, 3\}, \{\sigma_1, \sigma_2\}, M)$ be a semiautomaton whose transition graph is the following:



Semiautomaton A



Semiautomaton A_1



Semiautomaton A_2

$\rho_1=\{\{1, 3\}, \{2\}\}$ $\rho'_1=\{\{1\}, \{2, 3\}\}$

$\rho_2=\{\{1\}, \{2, 3\}\}$

ρ_1, ρ_2 are elementary in A_1 and ρ'_1 is elementary in A_2.

$\rho_{11}=\rho_1+\rho'_1=\{\{1, 2, 3\}\}$ $\tilde{\rho}_{11}=\{\{1, 2, 3\}\}$

$\rho_{21}=\rho_2+\rho'_1=\{\{1\}, \{2, 3\}\}$ $\tilde{\rho}_{21}=\{\{1\}, \{2, 3\}\}$

$\tilde{\rho}_{21}$ is elementary in A, but $\tilde{\rho}_{11}$ is not so.

Theorem 2. *Let ρ_i $(1 \leq i \leq n)$ be partitions such that $\rho_i \in \mathcal{P}_i$ and put $\rho=\sum_{i=1}^{n} \rho_i$. If there exist no partitions $\rho'_i \in \mathcal{P}_i$ $(1 \leq i \leq n)$ such that $\tilde{\rho}' < \tilde{\rho}$ $(\rho'=\sum_{i=1}^{n} \rho'_i)$, then $\tilde{\rho}$ is elementary in A.*

Proof. Suppose $\tilde{\rho}$ not to be elementary in A under the above assumption. Then, there exists an elementary partition π in A such that $0 < \pi < \tilde{\rho}$. In this case, from Theorem 1, there exist elementary partitions $\rho'_i \in \mathcal{P}_i$ $(1 \leq i \leq n)$ such that $\tilde{\rho}'=\pi$, where $\rho'=\sum_{i=1}^{n} \rho'_i$. Consequently, we get $0 < \tilde{\rho}'=\pi < \tilde{\rho}$. But, this is a contradiction. Q.E.D.

4. Computation of $\tilde{\pi}$. Let π be a partition of S. For each natural number p, we construct inductively the partition $\pi^{(p)}$ of S, starting with $\pi^{(0)}=\pi$. The construction method of $\pi^{(p)}$ $(p\geq 1)$ from $\pi^{(p-1)}$ is as follows:

Let $\pi^{(p-1)}=\{B_1, B_2, \cdots, B_m\}$ *be a partition of* S, *where each of* B_i*'s is a block of* $\pi^{(p-1)}$.

(i) *For each pair of numbers* s, t $(1\leq s\leq n, 1\leq t\leq m)$, *compute the set* $B_{st}=B_t\sigma_s^A$.

(ii) *For each pair of numbers* i, j $(1\leq i, j\leq m)$, *check whether* $B_i\sim B_j$, *according to the following definition*:
$B_i\sim B_j$ *if and only if* $B_i=B_j$, *or there exist some numbers* s, t $(1\leq s\leq n, 1\leq t\leq m)$ *such that* $B_i\cap B_{st}\neq\emptyset, B_j\cap B_{st}\neq\emptyset$.

(iii) *For each pair of numbers* i, j $(1\leq i, j\leq m)$, *check whether* $B_i\approx B_j$, *according to the following definition*:
$B_i\approx B_j$ *if and only if there exists some sequence of numbers* $i=i_0, i_1, i_2, \cdots, i_u=j$ *such that* $B_{i_w}\sim B_{i_{w+1}}$ $(w=0,1,2,\cdots,u-1)$.

(iv) *For each natural number* i $(1\leq i\leq m)$, *compute the set* $\bar{B}_i=\bigcup_{B_i\approx B_j}B_j$.

(v) *Let* $\pi^{(p)}$ *be the partition of* S *whose set of all blocks is* $\{\bar{B}_i; 1\leq i\leq m\}$.

From the following procedure, the admissible closure $\tilde{\pi}$ of π can be determined:

(vi) *Find a number* q *such that* $\pi^{(q)}=\pi^{(q=1)}$.

(vii) *Put* $\tilde{\pi}=\pi^{(q)}$.

5. Algorithm. We can now give the following algorithm for constructing all elementary partitions of the state set S in a semiautomaton $A=(S, \Sigma, M)$ $(\Sigma=\{\sigma_1, \sigma_2, \cdots, \sigma_n\})$:

(i) *For each natural number* i $(1\leq i\leq n)$, *construct the one-input semiautomaton* $A_i=(S, \Sigma_i, M_i)$.

(ii) *For each natural number* i $(1\leq i\leq n)$, *construct the set of all elementary partitions of* S *in* A_i, *i.e.*, $\mathcal{P}_i$.

(iii) *Construct the following set*:

$$\mathcal{P}=\{\rho; \rho=\sum_{i=1}^{n}\rho_i, \rho_i\in\mathcal{P}_i\}.$$

(iv) *Construct the following set*:

$$\tilde{\mathcal{P}}=\{\tilde{\rho}; \rho\in\mathcal{P}\}.$$

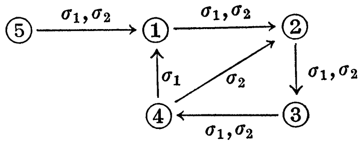
(v) *For each element* $\tilde{\rho}$ *in* $\tilde{\mathcal{P}}$, *construct the set* $\tilde{\mathcal{P}}(\tilde{\rho})=\{\tilde{\xi}; \tilde{\rho}<\tilde{\xi}, \tilde{\xi}\in\tilde{\mathcal{P}}\}$.

(vi) *Compute the following set*:

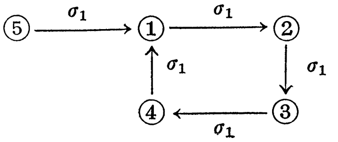
$$\mathcal{E}=\tilde{\mathcal{P}}-\bigcup_{\tilde{\rho}\in\tilde{\mathcal{P}}}\tilde{\mathcal{P}}(\tilde{\rho}).$$

(vii) $\mathcal{E}$ *forms the set of all elementary partitions of* S *in* A.

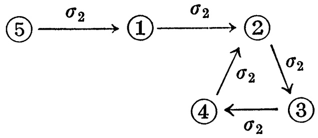
6. Example. Let $A=(\{1,2,3,4,5\}, \{\sigma_1, \sigma_2\}, M)$ be a semiautomaton whose transition graph is the following:



Semiautomaton A



Semiautomaton A_1



Semiautomaton A_2

$$\mathscr{P}_1=\{\rho_1, \rho_2\} \qquad \mathscr{P}_2=\{\rho_1', \rho_2'\}$$

$$\rho_1=\{\{1, 3\}, \{2, 4\}, \{5\}\} \qquad \rho_1'=\{\{1\}, \{2, 3, 4\}, \{5\}\}$$

$$\rho_2=\{\{1\}, \{2\}, \{3\}, \{4, 5\}\} \qquad \rho_2'=\{\{1, 4\}, \{2\}, \{3\}, \{5\}\}$$

$$\mathscr{P}=\{\rho_{11}, \rho_{12}, \rho_{21}, \rho_{22}\}$$

$$\rho_{11}=\rho_1+\rho_1'=\{\{1, 2, 3, 4\}, \{5\}\} \qquad \tilde{\rho}_{11}=\{\{1, 2, 3, 4\}, \{5\}\}$$

$$\rho_{12}=\rho_1+\rho_2'=\{\{1, 2, 3, 4\}\}, \{5\}\} \qquad \tilde{\rho}_{12}=\{\{1, 2, 3, 4\}, \{5\}\}$$

$$\rho_{21}=\rho_2+\rho_1'=\{\{1\}, \{2, 3, 4, 5\}\} \qquad \tilde{\rho}_{21}=\{\{1, 2, 3, 4, 5\}\}$$

$$\rho_{22}=\rho_2+\rho_2'=\{\{1, 4, 5\}, \{2\}, \{3\}\} \qquad \tilde{\rho}_{22}=\{\{1, 2, 3, 4, 5\}\}$$

$$\tilde{\mathscr{P}}=\{\{\{1, 2, 3, 4\}, \{5\}\}, \{\{1, 2, 3, 4, 5\}\}\}$$

$$\mathscr{E}=\{\{\{1, 2, 3, 4\}, \{5\}\}\}$$

Therefore, $\{\{1, 2, 3, 4\}, \{5\}\}$ is the unique elementary partition of $\{1, 2, 3, 4, 5\}$ in A.

References

[1] Ginzburg, A.: Algebraic Theory of Automata. Academic Press, New York—London (1968).

[2] Yoeli, M., and A. Ginzburg: On homomorphic images of transition graphs. J. Franklin Inst., **278**, 291–296 (1964).