155. *Incompleteness of Semantics for Intermediate Predicate Logics. I*

Kripke's Semantics

By Hiroakira ONO
Tsuda College

(Comm. by Kinjirô KUNUGI, M. J. A., Nov. 12, 1973)

Kripke models have been introduced in [2] for the intuitionistic logic, but in [3] we have studied their basic properties as models for intermediate propositional logics. We presented there the following problem.

Has every intermediate propositional logic a characteristic Kripke model?

Though it is important, e.g. in connection with the finite model property ([4]), it remains unsolved. Instead of this, we will show in the present paper that there exists an intermediate predicate logic having no characteristic Kripke models. In this sense, we can say that Kripke's semantics for intermediate predicate logics is incomplete. Similar incompleteness results for algebraic semantics will appear in the subsequent paper.

In order to abbreviate definitions, we use some of the terminology in Church [1]. We identify the word *predicate logics* with the word *pure functional calculi of first order.*

At first, we fix a language of pure functional calculus of first order. LK (and LJ) denote the pure classical (and intuitionistic) functional calculus of first order. The definition of intermediate predicate logics in general sense is given in [5], but here we only deal with finitely axiomatizable ones. Let A be a formula provable in LK. Then by $LJ+A$, we mean the intermediate predicate logic obtained by adding an axiom scheme A to LJ.

Definition 1. A pair (M, V) is called a Kripke model if M is a nonempty set with a partial order (denoted as $\leq$) and V a function from M to the power set of a set such that $V(a) \subseteq V(b)$ if $a \leq b$ and $V(a) \neq \emptyset$ for any $a \in M$.

A valuation W on (M, V) is a function which takes one of truth values $\{t, f\}$ as its value for a pair (A, a) of a formula A and an element a of M and whose values are determined by the rules in [2]. A formula A is said to be *valid* in (M, V), if $W(A^*, a) = t$ for any valuation W on (M, V) and any $a \in M$, where A^* is the closure of A. We write the set of formulas valid in (M, V) as $L(M, V)$. If L is an intermediate predicate logic and if there exists a Kripke model (M, V) such that $L(M, V)$ is equal to the set of formulas provable in L, then we say (M, V) is a characteristic Kripke model for L. In such a case, we say that L has a *characteristic Kripke model* (M, V).

We remark here that for any set $\{(M_i, V_i)\,; i \in I\}$ of Kripke models there exists a Kripke model (M, V) such that

$$L(M, V) = \bigcap_{i \in I} L(M_i, V_i).$$

Thus we can show that LJ, $LJ + (x)(A(x) \vee B) \supset (x)A(x) \vee B$ and $LJ + \neg\neg(x)(A(x) \vee \neg A(x))$ etc. have characteristic Kripke models.

Theorem. *There exists an intermediate predicate logic having no characteristic Kripke models.*

Proof. At first we show the following two lemmas.

Lemma 1. *Let (M, V) be any Kripke model. Then there exist Kripke models (M_i, V_i) $(i \in I)$ such that*

1) $L(M, V) = \bigcap_{i \in I} L(M_i, V_i)$,

2) *each M_i has the least element as a partially ordered set.*

This can be proved similarly as Theorem 2.10 in [3].

Lemma 2. *If M has the least element and the formula $(\exists x)(y)\neg\neg(A(x) \supset A(y))$ is vaid in (M, V), then V is a constant mapping.*

Proof. Suppose that V is not constant. Then there is an element a in M such that $V(0) \subsetneq V(a)$, where 0 is the least element of M. Define a valuation W on (M, V) by

$$W(A, b) = V(0) \qquad \text{for any } b \in M.$$

Let v be any element in $V(a) - V(0)$. Then for any u in $V(0)$, we can show that

$$W(\neg\neg(A(\bar{u}) \supset A(\bar{v})), a) = f,$$

where $\bar{u}$ and $\bar{v}$ denote the name of u and v, respectively. Thus,

$$W((y)\neg\neg(A(\bar{u}) \supset A(y)), 0) = f \qquad \text{for any } u \in V(0).$$

So,

$$W((\exists x)(y)\neg\neg(A(x) \supset A(y)), 0) = f.$$

This contradicts the hypothesis. Thus the lemma is proved.

Now we proceed to show the theorem. For brevity, we write

$$(x)(A(x) \vee B) \supset (x)A(x) \vee B \qquad \text{as } D \text{ and}$$

$$(\exists x)(y)\neg\neg(A(x) \supset A(y)) \qquad \text{as } E.$$

We will show that $LJ + E$ has no characteristic Kripke models. Suppose that $LJ + E$ has a characteristic Kripke model. Then by Lemmas 1 and 2, there exist Kripke models (M_i, V_i) $(i \in I)$ such that

1) the set of formulas provable in $LJ + E$ is equal to the set $\bigcap_{i \in I} L(M_i, V_i)$ and

2) each V_i is a constant mapping.

It is easy to verify that D is valid in (M_i, V_i) for each $i \in I$, since V_i is constant. Thus we have that D is provable in $LJ+E$ by 1). But Umezawa proved in [7] that D is not provable in $LJ+E$. So we have a contradiction. Thus the theorem is proved.

We notice that our theorem does not conflict with the strong completeness theorem of LJ in Thomason [6]. For, let (M, V) be the Kripke model constructed by the method in [6], by taking the set of formulas provable in $LJ+E$ for Γ and D for A. Then by [6] it follows that there is a valuation W such that $W(B, a)=t$ for any $a \in M$ and any B in Γ but $W(D, c)=f$ for some $c \in M$. However, we can show that E is not valid in (M, V).

References

[1] A. Church: Introduction to Mathematical Logic. Princeton Univ. Press, Princeton, N. J. (1956).

[2] S. A. Kripke: Semantical Analysis of Intuitionistic Logic. I (ed. J. Crossley and M. Dummett). Formal Systems and Recursive Functions. Amsterdam, pp. 92–129 (1965).

[3] H. Ono: Kripke models and intermediate logics. Publ. RIMS, Kyoto Univ., **6**, 461–476 (1970/71).

[4] ——: On the finite model property for Kripke models. Publ. RIMS, Kyoto Univ., **7**, 85–93 (1971/72).

[5] ——: A study of intermediate predicate logics. Publ. RIMS, Kyoto Univ., **8**, 619–649 (1972/73).

[6] R. H. Thomason: On the strong semantical completeness of the intuitionistic predicate calculus. J. Symbolic Logic, **33**, 1–7 (1968).

[7] T. Umezawa: On logics intermediate between intuitionistic and classical logic. J. Symbolic Logic, **24**, 141–153 (1959).