154. Estimates from $W_{p,\alpha}$ to $W_{q,\beta}$ for the Solutions of the Petrovskii Well Posed Cauchy Problems

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1. Introduction and results.

In this note, we shall consider the Cauchy problem

$$(1)\qquad \begin{cases} \dfrac{\partial u(t,x)}{\partial t} = P(D)u(t,x) & (t,x) \in (0,\infty)\times R^n, \\ u(0,x) = u_0(x) & x \in R^n. \end{cases}$$

Here $P(D)$ is the pseudo-differential operator of order d, that is,

$$(2)\qquad P(D)u = F^{-1}(S\hat{u}), \qquad u \in \mathcal{S}'^N,$$

where $S=(s_{ij})_{1\leqslant i,j\leqslant N}$ is the $N\times N$ matrix of functions s_{ij} in $C^\infty(R^n)$ which satisfy, for all multi-indices $\sigma=(\sigma_1,\cdots,\sigma_n)$,

$$(3)\qquad |D^\sigma s_{ij}(y)| \leqslant C_\sigma(1+|y|)^{d-|\sigma|}$$

where C_σ are constants depending on σ, $D^\sigma=(\partial/\partial y_1)^{\sigma_1}\cdots(\partial/\partial y_n)^{\sigma_n}$ and $|\sigma|=\sigma_1+\cdots+\sigma_n$. The matrix S will be called the symbol of P. In the above, $\mathcal{S}'^N$, F^{-1} and $\hat{u}$ denote the space of all N-tuples of distributions in the dual space $\mathcal{S}'$ of the Schwartz space $\mathcal{S}$, the inverse Fourier transformation and the Fourier transform of u, respectively. We assume that the order d of P is positive.

Let $\lambda_j(y)$ denote the eigenvalues of $S(y)$ for $j=1,2,\cdots,N$. We say that the Cauchy problem (1) is Petrovskii well posed if

$$(4)\qquad \mathrm{Re}\,\lambda_j(y) \leqslant \Lambda, \qquad 1\leqslant j\leqslant N,\ y\in R^n,$$

are valid for some constant Λ. When the Cauchy problem (1) is Petrovskii well posed, we can solve the problem in $\mathcal{S}'^N$ and the solution can be written as

$$(5)\qquad u(t)=E(t)u_0=F^{-1}(\exp(tS)\hat{u}_0) \qquad \text{for } u_0\in\mathcal{S}'^N.$$

We call the operator $E(t): u_0 \to u(t)$ the solution operator.

Let $1\leqslant p\leqslant\infty$. For $u\in L_p^N$ (the space of all N-tuples of functions in $L_p(R^n)$), we set

$$\|u\|_p = \begin{cases} \left(\displaystyle\int_{R^n} |u(x)|^p\,dx\right)^{1/p} & \text{if } p<\infty \\ \text{ess sup}\,\{|u(x)|\,;\ x\in R^n\} & \text{otherwise.} \end{cases}$$

For $\alpha\geqslant 0$, let $v_\alpha(y)=(1+|y|^2)^{\alpha/2}$ and

$$\|u\|_{p,\alpha}=\|F^{-1}(v_\alpha\hat{u})\|_p \qquad \text{for } u\in L_p^N.$$

We define $W_{p,\alpha}^N=\{u\in L_p^N\,;\ \|u\|_{p,\alpha}<\infty\}$.

Henceforth, for given p and q, we set $\gamma(p,q)=\max(1/2-1/p, 1/q-1/2, 0)$. Our results are the following.

Theorem 1. *Assume that the Cauchy problem* (1) *is Petrovskii well posed. Suppose that* $1\leqslant p\leqslant q\leqslant\infty$. *If*

$$\alpha-\beta>n(1/p-1/q)+nd\gamma(p,q)+(N-1)d,$$

then the inequality

$$(6)\qquad \|E(t)u_0\|_{q,\beta}\leqslant C(t)\|u_0\|_{p,\alpha},\qquad u_0\in W^N_{p,\alpha}$$

holds with some function $C(t)$ *which is bounded by a constant multiple of* $e^{At}(1+t)^{N-1+n\gamma(p,q)}$. *Moreover, if* $1<p\leqslant 2\leqslant q<\infty$, *then the inequality* (6) *is valid even when* $\alpha-\beta=n(1/p-1/q)+nd\gamma(p,q)+(N-1)d$.

Theorem 2. *If* $\alpha-\beta<n(1/p-1/q)+nd\gamma(p,q)+(N-1)d$, *then there exists a pseudo-differential operator* $P(D)$ *of order* d *for which the Cauchy problem* (1) *is Petrovskii well posed and the solution operator* $E(t)$ *is not bounded from* $W^N_{p,\alpha}$ *to* $W^N_{q,\beta}$ *for each* $t>0$. *Further, if* $d\neq 1$ *and if* $p=1$ *or* $q=\infty$, *then the same conclusion as above holds for* $\alpha-\beta=n(1/p-1/q)+nd\gamma(p,q)+(N-1)d$.

Remarks. Theorem 1 is a generalization of the results obtained by Sjöstrand [8] (for the Schrödinger equation) and the author [7] (for the case that $N=1$ and S is a pure imaginary polynomial function).

Considering L_p-L_q estimates for pseudo-differential operators, Hörmander has obtained the essentially same result as Theorems 1 and 2 for the case $d<1$ in [5].

2. Proof of Theorem 1.

We first define

$$M^N_{p,q}=M^N_{p,q}(R^n)=\{A=(a_{ij})_{1\leqslant i,j\leqslant N}\,;\,a_{ij}\in\mathcal{S}',\,M^N_{p,q}(A)<\infty\}$$

where

$$M^N_{p,q}(A)=\sup\{\|F^{-1}(A\hat{u})\|_p\,;\,u\in\mathcal{S}^N\text{ with }\|u\|_p=1\}.$$

When $N=1$, we merely write $M_{p,q}$ for $M^N_{p,q}$ and, in case $p=q$, we shall omit the subscript q of $M^N_{p,q}$. We refer to Hörmander [4] and Brenner [2] for the relevant facts about $M^N_{p,q}$.

The following Lemma 1 is fundamental.

Lemma 1. $A=(a_{ij})_{1\leqslant i,j\leqslant N}$ *belongs to* $M^N_{p,q}$ *if and only if* $a_{ij}\in M_{p,q}$ *for all* i,j, $1\leqslant i,j\leqslant N$. *Moreover, the inequality*

$$(7)\qquad c\,M^N_{p,q}(A)\leqslant\max\,(M_{p,q}(a_{ij})\,;\,1\leqslant i,j\leqslant N)\leqslant C\,M^N_{p,q}(A)$$

holds for some constants $c,C>0$.

The proof is easy and so we omit it.

We need two more lemmas to prove the theorem. For any $N\times N$ matrix A, $m(A)$ will denote the matrix norm, that is,

$$m(A)=\sup\{|Au|\,;\,u\in R^N,\,|u|=1\}.$$

Lemma 2 (Bernstein's theorem). *Let* $J=[n/2]+1$. *Let* $A=(a_{ij})$ *be a* $N\times N$ *matrix satisfying* $a_{ij}\in C^J(R^n)$ *for all* i,j, $1\leqslant i,j\leqslant N$. *Suppose that* $m(D^\sigma A)\in L_2(R^n)$ *for all* σ, $|\sigma|\leqslant J$. *Then,* $A\in M^N_1$ *and the inequality*

$$(8)\qquad M^N_1(A)\leqslant C\,\|m(A)\|_2^{1-n/(2J)}\Big(\sum_{|\sigma|=J}\|m(D^\sigma A)\|_2\Big)^{n/(2J)}$$

holds for some constant $C>0$.

Proof. By the usual Bernstein's theorem (see e.g. Sjöstrand [8]), we have

$$M_1(a_{ij})\leqslant C\,\|a_{ij}\|_2^{1-n/(2J)}\Big(\sum_{|\sigma|=J}\|D^\sigma a_{ij}\|_2\Big)^{n/(2J)}$$
$$\leqslant C\,\|m(A)\|_2^{1-n/(2J)}\Big(\sum_{|\sigma|=J}\|m(D^\sigma A)\|_2\Big)^{n/(2J)}.$$

Hence, by Lemma 1, we get

$$M_1^N(A)\leqslant C'\,\|m(A)\|_2^{1-n/(2J)}\Big(\sum_{|\sigma|=J}\|m(D^\sigma A)\|_2\Big)^{n/(2J)}.$$

This proves Lemma 2.

Lemma 3. *Let B be a $N\times N$ matrix and $\lambda_j, 1\leqslant j\leqslant N$, be eigenvalues of B. Set $\Lambda=\max(\mathrm{Re}\,\lambda_j\,;\,1\leqslant j\leqslant N)$. The following estimate holds:*

$$m(e^B)\leqslant e^\Lambda\sum_{j=0}^{N-1}(2m(B))^j. \tag{9}$$

For the proof of this lemma, we refer to Gelfand-Shilov [3].

Proof of Theorem 1. Without any loss of generality, we may assume $\beta=0$. Let us set $A(t,y)=(1+|y|^2)^{-\alpha/2}e^{tS(y)}$ for $(t,y)\in(0,\infty)\times R^n$. We shall show below that

$$M_{p,q}^N(A(t))\leqslant C(t), \tag{10}$$

which proves Theorem 1 when $p<\infty$. When $p=\infty$, Theorem 1 is a stronger assertion than (10) and we need a slight modification. For such a modification, see the author [7].

We divide our consideration into three cases. We first consider the case $1\leqslant p\leqslant 2\leqslant q\leqslant\infty$. When $p\neq 1$ and $q\neq\infty$, we set $\zeta=n(1/p-1/2)$ and $\eta=n(1/2-1/q)$. Putting $A'(t,y)=v_\zeta(y)A(t,y)v_\eta(y)$ for $y\in R^n$, we have

$$m(A'(t,y))\leqslant C_1C(t) \tag{11}$$

by Lemma 3 and the assumption on α, and hence $A'(t)\in M_2^N$.

By the Hardy-Littlewood-Sobolev theorem, we see that $v_{-\zeta}\in M_{p,2}^N$ and $v_{-\eta}\in M_{2,q}^N$. Therefore, $A(t)=v_{-\zeta}A'(t)v_{-\eta}\in M_{p,q}^N$ and

$$M_{p,q}^N(A(t))\leqslant C_2C(t). \tag{12}$$

By the assumption on α, it is possible to take $\zeta>n(1/p-1/2)$ (when $p=1$) and $\eta>n(1/2-1/q)$ (when $q=\infty$), so that the inequality (11) holds. So we can show (12) in the same manner as above in case $p=1$ or $q=\infty$.

Next we turn to the case $1\leqslant p\leqslant q<2$. We divide α into $\alpha=\alpha'+\alpha''$ where $\alpha'>nd(1/q-1/2)+(N-1)d$ and $\alpha''>n(1/p-1/q)$. Define $A''(t,y)=v_{-\alpha'}(y)e^{tS(y)}$ for $(t,y)\in(0,\infty)\times R^n$. Let us choose a function $\phi_0(r)$ in $C^\infty(R)$ which equals to 1 for $r<1$ and vanishes for $r>2$, and we put $\phi_k(r)=\phi_0(2^{-k}r)-\phi_0(2^{-k+1}r)$ for $k=1,2,\cdots$. We decompose A'' as $A''=\sum_{k=0}^\infty A_k''$, where $A_k''(t,y)=\phi_k(|y|)A''(t,y)$. By Lemma 3, we have

$$m(e^{tS(y)})\leqslant Ce^{At}(1+t)^{N-1}(1+|y|)^{(N-1)d}.$$

Using this estimate and (3), we easily get

$$m(D^\sigma A_k''(t,y))\leqslant Ce^{At}(1+t)^{|\sigma|+N-1}2^{(d-1)k|\sigma|-\alpha'k+(N-1)dk}.$$

Hence, by Lemma 2,

$$M_1^N(A_k''(t))\leqslant Ce^{At}(1+t)^{n/2+(N-1)}2^{nkd/2-\alpha'k+(N-1)dk}.$$

It is easy to see from Lemmas 1 and 3 that

$$M_2^N(A_k''(t))\leqslant C\,\|m(A_k''(t))\|_\infty\leqslant C'e^{At}(1+t)^{N-1}2^{(N-1)dk-\alpha'k}.$$

Applying the Riesz-Thorin's convexity theorem, we obtain

$$M_q^N(A_k''(t))\leqslant Ce^{At}(1+t)^{N-1+n(1/q-1/2)}2^{(N-1)dk-\alpha'k+n(1/q-1/2)k}.$$

Summing over all k, we have

$$M_q^N(A''(t))\leqslant\sum_{k=0}^{\infty}M_q^N(A_k''(t))\leqslant Ce^{At}(1+t)^{N-1+n(1/q-1/2)}.$$

On the other hand, by the Hardy-Littlewood-Sobolev theorem, we get $v_{-\alpha''}\in M_{p,q}^N$. Therefore, we have

$$M_{p,q}^N(A(t))\leqslant C(t).$$

In case $2<p\leqslant q\leqslant\infty$, our theorem is easily shown by the standard duality argument. This finishes the proof.

3. Proof of Theorem 2.

We begin with the well-known lemma.

Lemma 4. *Set* $v_\delta(y)=(1+|y|^2)^{\delta/2}$ *for* $y\in R^n$. *If* $\delta>-n(1/p-1/q)$ *and* $1\leqslant p\leqslant q\leqslant\infty$, *then* $v_\delta\notin M_{p,q}$. *Moreover, in case* $p=1$ *or* $q=\infty$, $v_\delta\notin M_{p,q}$ *for* $\delta=-n(1/p-1/q)$.

For the proof, see Stein [9].

The next lemma was proved by Wainger [10] ($0<d<1$ and $p=q$), Hörmander [5] ($0<d<1$) and Sjöstrand [8] ($d>1$ and $p=q$). Here and later the letter ψ denotes a function in $C^\infty(R)$ satisfying $\psi(r)=1$ for $r>2$ and $\psi(r)=0$ for $r<1$, and let $w_\delta(y)=\psi(|y|)\,|y|^{-\delta}\exp(i|y|^d)$ for $y\in R^n$ and $d>0$.

Lemma 5. *If* $d\neq1$, $1\leqslant p\leqslant q<2$ *and* $\delta<n(1/p-1/q)+nd(1/q-1/2)$, *then* $w_\delta\notin M_{p,q}(R^n)$. *Especially, if* $p=1$, *then* $w_\delta\notin M_{p,q}$ *for* $\delta=n(1/p-1/q)+nd(1/q-1/2)$.

Proof. First we assume $\delta<n(1/p-1/q)+nd(1/q-1/2)$. Let $p'=p/(p-1)$ and $\hat{g}(y)=\psi(|y|)\,|y|^{-\theta}$ with $\theta=n/p'+n(1/p-1/q)+nd(1/q-1/2)-\delta$. We know that $g\in L_p(R^n)$ since $\theta>n/p'$ (see Sjöstrand [8]). Putting $\hat{f}(y)=w_\delta(y)\hat{g}(y)$ for $y\in R^n$, the asymptotic behavior of f is as follows: (i) If $d>1$, then

(13) $$|f(x)|=C_{d,\delta+\theta}|x|^{(n-\delta-\theta-nd/2)/(d-1)}+O(|x|^\omega)$$

as $|x|\to\infty$, where $\omega<(n-\delta-\theta-nd/2)/(d-1)$ and where $C_{d,\delta+\theta}$ is a positive constant.

(ii) If $d<1$, then

(14) $$|f(x)|=C_{d,\delta+\theta}|x|^{(n-\delta-\theta-nd/2)/(d-1)}+O(|x|^\omega)$$

as $|x|\to0$, where $\omega>(n-\delta-\theta-nd/2)/(d-1)$ and where $C_{d,\delta+\theta}$ is a positive constant.

Since $q(n-\delta-\theta-nd/2)/(d-1)=-n$, f does not belong to $L_p(R^n)$. This means $w_\delta \notin M_{p,q}(R^n)$.

We turn to the case $p=1$ and $\delta=n(1/p-1/q)+nd(1/q-1/2)$. It is well-known (see Hörmander [4]) that

$$M_{1,q}=FL_q \text{ for } q>1 \text{ and } M_1=FM.$$

Here FL_q denotes the space of all Fourier transforms of functions in L_q and FM denotes the space of all Fourier-Stieltjes transforms of bounded measures.

On the other hand, the inverse Fourier transform of the function w_δ is asymptotically described by the right hand side of (13) or (14) with $\theta=0$. So we have $w_\delta \notin M_{1,q}$ since $q(n-\delta-nd/2)/(d-1)=-n$. Thus we have proved Lemma 5.

Proof of Theorem 2. Seeing that $M^N_{p,q}=\{0\}$ for $p>q$ (see Hörmander [4]), we assume below that $p\leqslant q$. We may also assume $\beta=0$. The proof will be divided into three cases.

We first treat the case $p\leqslant 2\leqslant q$. Let us define the $N\times N$ matrix S by

$$S(y)=(1+|y|^2)^{d/2}\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix} \quad \text{for } y\in R^n. \tag{15}$$

The $(1,N)$ element of $e^{tS(y)}$ is given by

$$\frac{1}{(N-1)!}(1+|y|^2)^{(N-1)d/2}t^{N-1}.$$

Therefore, in view of Lemmas 1 and 4, we see that

$$(1+|y|^2)^{-\alpha/2}e^{tS(y)} \notin M^N_{p,q}.$$

It is now easily checked that the pseudo-differential operator $P(D)$ defined by (2) satisfies the desired properties.

We turn to the case $d\neq 1$ and $1\leqslant p\leqslant q<2$ (or $2<p\leqslant q\leqslant\infty$). Let us set

$$S(y)=i\psi(|y|)|y|^d\begin{pmatrix} 1 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 1 \end{pmatrix},\ y\in R^n, \tag{16}$$

where the matrix is $N\times N$. Then, the $(1,N)$ element of the matrix function $e^{tS(y)}$ is

$$\frac{i^{N-1}}{(N-1)!}\psi(|y|)^{N-1}|y|^{N-1}\exp(i\psi(|y|)|y|^d)t^{N-1}.$$

When $1\leqslant p\leqslant q<2$, by Lemma 5, we easily see that the $(1,N)$ element of $(1+|y|^2)^{-\alpha/2}e^{tS(y)}$ does not belong to $M^N_{p,q}$. When $2<p\leqslant q\leqslant\infty$, it is shown by the duality argument that the same is also true. It follows from Lemma 1 that $(1+|y|^2)^{-\alpha/2}e^{tS(y)} \notin M^N_{p,q}$.

In case when $d=1$ and $1\leqslant p\leqslant q<2$ (or $2<p\leqslant q\leqslant\infty$), we construct the pseudo-differential operator having the required properties for each p, q and α. For fixed p, q and α satisfying $\alpha<n(1/p-1/q)+n\gamma(p,q)+N-1$, we choose a number d' which is smaller than 1 and satisfies $\alpha<n(1/p-1/q)+nd'\gamma(p,q)+(N-1)d'$. Then, the symbol S givin by (16) replaced d by d' defines the pseudo-differential operator having the required properties. The proof is completed.

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