# 148. *On Normalizers of Simple Ring Extensions*

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Throughout the present note, $A$ will represent an (Artinian) simple ring with the center $C$, and $B$ a regular subring of $A$ with the center $Z$. Let $V$ be the centralizer $V_A(B)$ of $B$ in $A$, and $N$ the normalizer $N_A(B)=\{a \in A^\cdot \mid B\tilde{a}=B\}$ of $B$ in $A$. As is well-known, $B_0=BV=B\otimes_Z V$ is two-sided simple. Obviously, $N\subseteq N_A(V)$ and $B^\cdot V^\cdot$ is a normal subgroup of $N$. We fix here a complete representative system $\{u_\lambda \mid \lambda \in \Lambda\}$ of $N$ modulo $B^\cdot V^\cdot$. As to notations and terminologies used without mention, we follow [2].

In case $A\neq (GF(2))_2$, it is known that if $N=A^\cdot$ then either $B=A$ or $B\subseteq C$ (see for instance [2; Proposition 8.10 (a)]). In what follows, we shall prove further results concerning $N$ such as P. Van Praag [1] obtained for division ring extensions.

**Lemma.** *The ring $BN=\sum_{u\in N} Bu$ is a completely reducible $B$-$B$-module with homogeneous components $B_0u_\lambda (\lambda \in \Lambda)$. Furthermore, every irreducible $B_0$-$B_0$-module $B_0u_\lambda$ is not isomorphic to $B_0u_\mu$ for $\mu\neq\lambda$.*

**Proof.** It is obvious that every $Bu (u \in N)$ is $B$-$B$-irreducible. Now, assume that $Bu$ is $B$-$B$-isomorphic to $Bu_\lambda$ and $u\leftrightarrow bu_\lambda (b \in B)$. Since $Bb=B$, $b$ is a unit of $B$. For every $b' \in B$, we have $ub'\leftrightarrow bu_\lambda b'=b\cdot b'\tilde{u}_\lambda \cdot u_\lambda$ and $b'\tilde{u}\cdot u \leftrightarrow b'\tilde{u}\cdot bu_\lambda$, and so $b\cdot b'\tilde{u}_\lambda = b'\tilde{u}\cdot b$, whence it follows $B\mid \widetilde{bu_\lambda}=B\mid \tilde{u}$. Hence, we obtain $(bu_\lambda)^{-1}u \in V^\cdot$, which implies that $u\in B^\cdot V^\cdot u_\lambda$. Conversely, every $Bvu_\lambda (v \in V^\cdot)$ is $B$-$B$-isomorphic to $Bu_\lambda$, and hence we have seen that $\bigoplus_{\lambda\in\Lambda} B_0u_\lambda$ is the idealistic decomposition of the $B$-$B$-module $BN$. Finally, if $B_0u_\lambda$ is $B_0$-$B_0$-isomorphic to $B_0u_\mu$ $(\mu\neq\lambda)$ then they are $B$-$B$-isomorphic, which yields a contradiction.

**Corollary.** *If $V\subseteq B$ then $BN$ is the direct sum of non-isomorphic irreducible $B$-$B$-submodules, and conversely.*

**Proposition 1.** *Assume that $BN=A$.*

(1) $[A:B]_L=[A:B]_R=(N:B^\cdot V^\cdot)[V:Z]$.

(2) *If $N'$ is a subgroup of $N$ containing $B^\cdot V^\cdot$ then $BN'\cap N=N'$.*

(3) *If $A'$ is a simple intermediate ring of $A/B_0$ then $A'=BN_{A'}(B)$.*

(4) *$V/C$ is Galois.*

**Proof.** (1) is clear by Lemma.

(2) By Lemma, $BN'=\bigoplus_{\lambda\in\Lambda'} B_0u_\lambda$ with a suitable subset $\Lambda'$ of $\Lambda$.

Then, as is easily seen, $N' = \bigcup_{\lambda \in \Lambda'} B^{\cdot}V^{\cdot}u_\lambda = BN' \cap N$.

(3) Again by Lemma, $A' = \bigoplus_{\lambda \in \Lambda'} B_0 u_\lambda$ with a suitable subset $\Lambda'$ of $\Lambda$. Since $A'$ is simple, we have then $N_{A'}(B) = A' \cap N = \bigcup_{\lambda \in \Lambda'} B^{\cdot}V^{\cdot}u_\lambda$, whence it follows $A' = BN_{A'}(B)$.

(4) Since $B$ is generated by its units, we obtain $J(V|\tilde{N}) = V_V(N) = V_V(BN) = C$.

Now, we are at the position to prove our theorem.

**Theorem.** *Let $A/B$ be a right locally finite extension such that $BN = A$.*

(1) *Every intermediate ring $A'$ of $A/B_0$ is simple and $A'$-$B_0$-irreducible, and there holds $[A' : B]_L = [A' : B]_R = (N_{A'}(B) : B^{\cdot}V^{\cdot})[V : Z]$.*

(2) *Let $N'$ be a subgroup of $N$ containing $B^{\cdot}V^{\cdot}$. If $BN' \cdot N_A(BN') = A$ then $N'$ is a normal subgroup of $N$, and conversely.*

(3) *$N' \mapsto BN'$ and $A' \mapsto N_{A'}(B)$ are mutually converse 1–1 correspondences between the set of subgroups $N'$ of $N$ containing $B^{\cdot}V^{\cdot}$ and the set of intermediate rings $A'$ of $A/B_0$.*

**Proof.** (1) By [2; Corollary 4.5], $B_0$ is a simple ring. Given a finite subset $F$ of $A$, there exists a finite subset $\Lambda'$ of $\Lambda$ such that $B[F] \subseteq \bigoplus_{\lambda \in \Lambda'} B_0 u_\lambda$. Again by the right local finiteness of $A/B$, we can find a finite subset $\Lambda''$ of $\Lambda$ such that $B[\{u_\lambda | \lambda \in \Lambda'\}] \subseteq \bigoplus_{\lambda \in \Lambda''} B_0 u_\lambda$. Obviously, $B_0[F] \subseteq B_0[\{u_\lambda | \lambda \in \Lambda'\}] \subseteq \bigoplus_{\lambda \in \Lambda''} B_0 u_\lambda$, which implies that $A/B_0$ is (left and) right locally finite. Next, if $M$ is an arbitrary non-zero $A$-$B_0$-submodule of $A$ then there exists some $\lambda$ such that $B_0 u_\lambda \subseteq M$(Lemma), which means $M = A$. Then, by [2; Proposition 3.8 (b)], $A'$ is a simple ring. Noting that $V_{A'}(B) = V$ and $A' = BN_{A'}(B)$ (Proposition 1 (3)), the other assertions are consequences of the fact mentioned just above and Proposition 1 (1).

(2) Let $A' = BN'$. Then, $V_A(A')$ is a subfield of the center of $V$. As was shown in (1), $A'$ is a simple intermediate ring of $A/B_0$ and $A'$-$B_0$-irreducible. Now, let $\{u'_\kappa | \kappa \in K\}$ be a complete representative system of $N_A(A')$ modulo $A'^{\cdot} = A'^{\cdot} V_A(A')^{\cdot}$ Then, by Lemma, we have $A = \bigoplus_{\kappa \in K} A' u'_\kappa$. We claim here that the last decomposition is the idealistic decomposition of $A$ as $A'$-$B_0$-module, too. In fact, every $A'u'_\kappa$ is $A'$-$B_0$-irreducible. If $A'u'_\kappa$ is $A'$-$B_0$-isomorphic to $A'u'_\nu$ and $u'_\kappa \leftrightarrow a'u'_\nu (a' \in A'^{\cdot})$, then the argument used in the proof of Lemma enables us to see that $(a'u'_\nu)^{-1} u'_\kappa \in V_A(B_0)^{\cdot} \subseteq A'^{\cdot}$. This means that $u'_\kappa \in A'^{\cdot} u'_\nu$, namely, $\nu = \kappa$. Now, let $u$ be an arbitrary element of $N$. Since $A'u$ is an irreducible $A'$-$B_0$-submodule of $A$, the last remark proves that $A'u = A'u'_\kappa$ for some $\kappa$. We have seen thus $N \subseteq N_A(A')$. Now, by Proposition 1 (2), $N' = N \cap A'$

$=N\cap A'^{\cdot}$, which is obviously a normal subgroup of $N$. The converse is almost evident.

(3) This is only a combination of (1) and Proposition 1 (2) and (3).

Even the following corollary contains all the main results in [1].

**Corollary.** *Let $A/B$ be a right locally finite extension such that $V\subseteq B$ and $BN=A$.*

(1) *Every intermediate ring $A'$ of $A/B$ is simple, and there holds $[A':B]_L=[A':B]_R=(N_{A'}(B):B^{\cdot})$.*

(2) *Let $N'$ be a subgroup of $N$ containing $B^{\cdot}$. If $BN'\cdot N_A(BN')=A$ then $N'$ is a normal subgroup of $N$, and conversely.*

(3) *$N'\mapsto BN'$ and $A'\mapsto N_{A'}(B)$ are mutually converse 1–1 correspondences between the set of subgroups $N'$ of $N$ containing $B^{\cdot}$ and the set of intermediate rings $A'$ of $A/B$.*

Finally, we state the following:

**Proposition 2.** *Assume that $[A:C]<\infty$.*

(1) *If $V\subseteq B$ then $(N:B^{\cdot})<\infty$, and the converse is true provided $V$ is infinite.*

(2) *Assume that $V\subseteq B$. If $BN=A$ then $V/C$ is Galois, and conversely.*

**Proof.** (1) Since $C\subseteq V\subseteq B$, it is well-known that $B=V_A(V)$, whence it follows $N=N_A(V)$. The mapping $f: N\to\mathfrak{G}(V, V; C)$ defined by $u\mapsto V|\tilde{u}^{-1}$ is a group homomorphism and $\operatorname{Ker} f=V_N(V)=B^{\cdot}$. Hence, $N/B^{\cdot}$ is isomorphic to a subgroup of the finite group $\mathfrak{G}(V, V; C)$, which yields $(N:B^{\cdot})<\infty$. Conversely, if $(N:B^{\cdot})<\infty$ then $\infty>(B^{\cdot}V^{\cdot}:B^{\cdot})=(V^{\cdot}:B^{\cdot}\cap V^{\cdot})=(V^{\cdot}:Z^{\cdot})$. Now, under the supplementary assumption that $V$ is infinite, we have $V=Z$ by [2; Lemma 3.9].

(2) Since $A/B$ is finite (inner) Galois and $V$ coincides with the field $Z$, it is known that every intermediate ring of $A/B$ is simple ([2; Theorem 7.3 (b)]). Now, noting that $V_A(BN)=C$ if and only if $BN=A$, our assertion is obvious by the proof of Proposition 1 (4).

## References

[1] P. Van Praag: Groupes multiplicatifs des corps. Bull. Soc. Math. Belgique, 23, 506–512 (1971).

[2] H. Tominaga and T. Nagahara: Galois Theory of Simple Rings. Okayama Math. Lectures (1970).