# 147. *Wave Equation with Wentzell's Boundary Condition and a Related Semigroup on the Boundary. I*

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1. For *diffusion equation* in one dimension, W. Feller [1] determined all types of possible boundary conditions. A part of the result was extended by A. D. Wantzell [2] for multi-dimensional case. In fact, he found a *candidate* for the most general boundary condition which is possible for diffusion equation

$$\frac{\partial u}{\partial t}=Au, \tag{1}$$

where $A$ is a second order elliptic differential operator on a compact domain $\bar{D}$ in $R^N$. More precisely, he proved that, under regularity conditions on $A$ and $D$, any smooth function $u$ in the domain of the generator, which is a contraction of $\bar{A}$, of a strongly continuous semigroup $\{T_t, t\geqslant 0\}$ of non-negative linear operators on $C(\bar{D})$ with norm $\|T_t\|\leqslant 1$,[1] necessarily satisfies a boundary condition:

$$Lu(x)=0, \qquad x\in\partial D.$$

$$\begin{aligned} Lu(x)=&\sum_{i,j=1}^{N-1}\alpha_{ij}(x)\frac{\partial^2 u}{\partial\xi_x^i\partial\xi_x^j}(x)+\sum_{i=1}^{N-1}\beta_i(x)\frac{\partial u}{\partial\xi_x^i}(x)\\ &+\gamma(x)u(x)+\delta(x)Au(x)+\mu(x)\frac{\partial u}{\partial n}(x)\\ &+\int_{\bar{D}}\left\{u(y)-u(x)-\sum_{i=1}^{N-1}\frac{\partial u}{\partial\xi_x^i}(x)\xi_x^i(y)\right\}\nu(x,dy),\end{aligned} \tag{2}$$

where $\{\alpha_{ij}(x)\}$ is symmetric and non-negative definite, $\gamma(x)$, $\delta(x)$, $-\mu(x)$ are non-positive, and $\nu(x,\cdot)$ is a measure on $\bar{D}$. $\{\xi_x^i(y), 1\leqslant i\leqslant N\}$ is a system of functions in $C^2(\bar{D})$ and is a local coordinate in a neighbourhood of $x$, and $(\partial u/\partial\xi_x^N)(x)$ coincides with the inner normal derivative $(\partial u/\partial n)(x)$.[2] We sometimes omit the suffix $x$ of $\xi_x^i$. Wentzell also proved that the boundary condition (2) is actually possible in an important special case. For the problem to solve (1) with his boundary condition in general case, we considered a method in [3], [4], which reduces the problem to solve an integro-differential equation on the

1) Here, the domain of the definition of $A$ is $C^2(\bar{D})$ and $\bar{A}$ is the closure of $A$ in $C(\bar{D})$.

2) For a more detailed information about the terms in $Lu(x)$, the reader can consult [2].

boundary.[3] Recently, the result was extended by Bony and others [5] along the same line.

Now, it seems to be of some interest to know whether Wentzell's boundary condition is also possible for the *wave equation*

$$\frac{\partial^2}{\partial t^2}u = Au, \tag{3}$$

and to know what is the intuitive meanings of non-classical terms of the boundary condition, if the answer to the first question is in the affirmative. In fact, Feller [6] already solved a one-dimensional wave equation

$$\frac{\partial^2}{\partial t^2}u = \Omega u = \frac{\partial}{\partial m}\frac{\partial^+}{\partial s}u,^{4)} \qquad x \in (r_1, r_2),$$

with boundary condition

$$\gamma_i \cdot u(r_i) + \delta_i \Omega u(r_i) + \frac{\partial}{\partial n}u(r_i) = 0, \qquad i = 1, 2,$$

and explained $\delta_i$ to be the effect of point masses attached to end points $r_i$ of a string between $r_1$ and $r_2$.

Here, we construct a semigroup solution of (2)–(3) in a Hilbert space set up, where $A$ and $L$ are restricted to a typical special case for simplicity. A result for a general case will be published elsewhere with a detailed proof. The method is an extension of K. Yosida [8], where he solved a wave equation on $R^N$ with uniformly elliptic $A$.[5]

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**2. An abstract scheme.** Let $\mathcal{H}_0$ be a real vector space with an inner product $(f, g)_s$, a norm $\|f\|_l$ and a positive constant $c$ such that

$$c\|f\|_s \leqslant \|f\|_l, \qquad \text{for } f \in \mathcal{H}_0, \tag{4}$$

where $\|f\|_s = (f, f)_s^{1/2}$. Let $\mathcal{H}$ and $\mathcal{K}$ be the completions of $\mathcal{H}_0$ with respect to the norms $\|f\|_s$ and $\|f\|_l$, respectively. We consider assumptions

(A. 1) $\mathcal{K}$ can be imbedded uniquely in $\mathcal{H}$.

(A. 2) There is a strongly continuous semigroup $\{T_t, t \geqslant 0\}$ of linear operators on $\mathcal{H}$ with norm $\|T_t\| \leqslant e^{bt}$, such that the domain $\mathcal{D} = \mathcal{D}(\mathcal{A})$ of the generator $\mathcal{A}$ is contained in $\mathcal{K}$.

(A. 3) $\mathcal{D}$ is a dense subset of $\mathcal{K}$.

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3) This method goes back to an idea of Feller [1], where $D$ is an interval and the integro-differential equation on $\partial D$ here is reduced to a pair of linear equations with two unknowns.

4) This is the intrinsic form of one-dimensional diffusion operator discovered by Feller [7].

5) As for the diffusion semigroup, related results on the Hilbert space set up have been obtained by H. Kunita [9] and M. Fukushima [10], [11].

(A. 4) For a constant $\alpha_0$, $(\alpha_0 f - \mathcal{A}f, f)_s = \|f\|_l^2$, for $f \in \mathcal{D}$.

(A. 5) For a constant $K$, $|(\mathcal{A}f, g)_s - (\mathcal{A}g, f)_s| \leqslant K(\|f\|_l^2 + \|g\|_s^2)$ for $f, g \in \mathcal{D}$.

**Theorem 1.** *Under the assumptions* (A. 1)–(A. 5), *let* $\boldsymbol{B}$ *be the direct product space* $\binom{\mathcal{K}}{\mathcal{H}}$ *with norm* $\left\|\binom{f}{g}\right\| = (\|f\|_l^2 + \|g\|_s^2)^{1/2}$, *and define* $\boldsymbol{G}$ *by*

$$\boldsymbol{G}\binom{f}{g} = \binom{g}{\mathcal{A}f}, \qquad for \binom{f}{g} \in \binom{\mathcal{D}}{\mathcal{K}}.$$

*Then,* $\boldsymbol{G}$ *is the generator of a strongly continuous group* $\{U_t, -\infty < t < \infty\}$ *of linear operators on* $\boldsymbol{B}$ *with norm* $\|U_t\| \leqslant e^{b't}$.

**Corollary.** *If we write* $\binom{u(t)}{v(t)} = U_t\binom{f}{g}$ *for* $\binom{f}{g} \in \binom{\mathcal{D}}{\mathcal{K}}$, *then*

$$(5) \qquad \frac{d^2}{dt^2}u = \mathcal{A}u, \qquad \lim_{t\to 0} u(t) = f \text{ in } \mathcal{K}, \ \lim_{t\to 0} v(t) = g \text{ in } \mathcal{H}.$$

**Remark.** (A. 1), (A. 2) *and the following conditions imply* (A. 3).

$$(6) \qquad \begin{aligned} &\alpha\|(\alpha - A)^{-1}f\|_l \leqslant \|f\|_l, && for\ f \in \mathcal{H}_0,\ \alpha \geqslant \alpha_0, \\ &\|\alpha(\alpha - A)^{-1}f - f\|_l \to 0 && as\ \alpha \to \infty,\ for\ f \in \mathcal{H}_0 \end{aligned}$$

**3. Construction of the solution.** Let $D$ be a bounded domain with smooth boundary in $R^N$, and let $\mathcal{H}_0$ be the set of all $M$-times continuously differentiable functions on $\bar{D} = D \cup \partial D$ for some fixed $M \geqslant 3$. We assume

$$Au(x) = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}(x),$$

$$(2') \qquad \begin{aligned} Lu(x) = a \sum_{i=1}^{N-1} \frac{\partial^2 u}{\partial \xi_i^2}(x) + \delta(x) Au(x) \\ + \frac{\partial u}{\partial n}(x) + \int_{\partial D} (u(y) - u(x))\nu(x, dy), \end{aligned}$$

where $a$ is a non-negative constant and $\delta(x)$ is smooth and non-positive. $\nu(x, dy)$ is concentrated on $\partial D - \{x\}$ and has a symmetric density $\nu(x, y)$ with respect to the surface element $dy$ on $\partial D$, such that

$$u \to \nu[u] = \int_{\partial D} (u(y) - u(x))\nu(x, dy)$$

maps $C^2(\bar{D})$ into $C(\partial D)$, and

$$(7) \qquad \begin{aligned} &\nu_\varepsilon = \sup_{|x-y| \geqslant \varepsilon} \nu(x, y) < \infty, && \text{for } \varepsilon > 0, \\ &\int_{\partial D} \sum_{i=1}^{N=1} |\xi_x^i(y)| \nu(x, dy) < \infty, && \text{for } x \in \partial D. \end{aligned}$$

For real valued functions $f$ and $g$, we write

$$(f, g) = \int_D f(x)g(x)dx, \qquad D(f, g) = \int_D \sum_1^N \frac{\partial f}{\partial x_i}(x)\frac{\partial g}{\partial x_i}(x)dx,$$

$$\langle f, g\rangle = \int_{\partial D} f(x)g(x)dx, \qquad D\langle f, g\rangle = \int_{\partial D} \sum_1^{N-1} \frac{\partial f}{\partial \xi_i}(x)\frac{\partial g}{\partial \xi_i}(x)dx,$$

$$\nu(f,g)=\frac{1}{2}\int_{\partial D}dx\int_{\partial D}(f(y)-f(x))(g(y)-g(x))\nu(x,dy),$$

where $dx$ is the volume element in case of the integrals on $D$. We define

$$(f,g)_s=(f,g)+\langle f,g\cdot|\delta|\rangle,\ \text{for } f,g\in\mathcal{H}_0,\quad \|f\|_s=(f,f)_s^{1/2},$$
$$B_\alpha(f,g)=\alpha(f,g)_s+D(f,g)+a\cdot D\langle f,g\rangle+\nu(f,g),\ \text{for } f,g\in\mathcal{H}_0,\ \alpha\geqslant 0,$$
$$(f,g)_l=B_1(f,g),\ \text{for } f,g\in\mathcal{H}_0,\quad \|f\|_l=(f,f)_l^{1/2}.\text{[6)]}$$

Let $\mathcal{H}$ and $\mathcal{K}$ be the completions of $\mathcal{H}_0$ with respect to $\|f\|_s$ and $\|f\|_l$, respectively.

**Proposition 1.** *$B_\alpha(f,g)$ can be extended uniquely to a bilinear functional on $\mathcal{K}$. The extension, under the same notation, satisfies*

(8) $|B_\alpha(f,g)|\leqslant c_\alpha\|f\|_l\cdot\|g\|_l,\ c'_\alpha\|f\|_l^2\leqslant B_\alpha(f,f),$ *for* $f,g\in\mathcal{K},\ \alpha>0.$

**Proposition 2.** *For $f$ and $g$ in $\mathcal{H}_0$,*

(9) $$((\alpha-A)f,g)_s-\langle Lf,g\rangle=B_\alpha(f,g),\ \alpha\geqslant 0.$$

This is proved by the Green-Stokes formulas for $\bar{D}$ and $\partial D$

$$(Af,g)+D(f,g)+\left\langle\frac{\partial}{\partial n}f,g\right\rangle=0,\qquad \left\langle\sum_1^{N-1}\frac{\partial^2 f}{\partial\xi_i^2},g\right\rangle+D\langle f,g\rangle=0$$

and $\langle\nu[f],g\rangle=-\nu(f,g)$, which is implied by the symmetry of $\nu(x,y)$.

**Lemma 1.** *If $\{f_n, n=1,2,\cdots\}$ in $\mathcal{H}_0$ is a Cauchy sequence in $\mathcal{K}$ such that $\lim_{n\to\infty}\|f_n\|_s=0$, then $\lim_{n\to\infty}\|f_n\|_l=0$. Hence, $\mathcal{K}$ is imbedded uniquely in $\mathcal{H}$.*

In case $\delta(x)=1$, $\|f\|_s=\{(f,f)+\langle f,f\rangle\}^{1/2}$, and hence we have

**Corollary.** *$\mathcal{K}$ is a set of real valued functions defined on $\bar{D}$.*

**Proposition 3.** *If $\{f_n, n=1,2,\cdots\}$ in $\mathcal{H}_0$ and $g\in\mathcal{H}$ satisfy $\lim_{n\to\infty}\|f_n\|_l=0$ and $\lim_{n\to\infty}\{(Af_n-g,h)_s+\langle Lf_n,h\rangle\}=0$ for each $h\in\mathcal{H}_0$, then $g=0$.*

**Definition.** If, for $f$ in $\mathcal{K}$, there are a sequence $\{f_n, n=1,2,\cdots\}$ in $\mathcal{H}_0$ and $g\in\mathcal{H}$ such that $\lim_{n\to\infty}\|f_n-f\|_l=0$ and

$$\lim_{n\to\infty}\{(Af_n-g,h)_s+\langle Lf_n,h\rangle\}=0\qquad\text{for each } h\in\mathcal{H}_0,$$

then we define $\bar{A}_Lf=g$, and denote the set of all such $f$ by $\mathcal{D}(\bar{A}_L)$.[7)]

**Proposition 4.**

(10) $$(\alpha f-\bar{A}_Lf,g)_s=B_\alpha(f,g)\qquad \textit{for } f\in\mathcal{D}(\bar{A}_L),\ g\in\mathcal{K}$$
$$\alpha f-\bar{A}_Lf=0,\qquad \textit{implies } f=0$$

**Proposition 5.** *For each $v\in\mathcal{K}$,*

(11) $$\alpha u-\bar{A}_Lu=v,\qquad \alpha>0$$

*has a unique solution $u$, which satisfies*

(12) $$B_\alpha(u,f)=(v,f)_s\ \textit{for each } f\in\mathcal{K},\qquad c'_\alpha\|u\|_l\leqslant\|v\|_s.$$

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6) In general case, we consider non-symmetric $B_\alpha(f,g)$ and a Hilbert space $\mathcal{K}$ whose norm is equivalent with $\|f\|_l=B_{\alpha_0}(f,f)^{1/2}$ for a sufficiently large $\alpha_0$. But, the results and the proofs are essentially the same.

7) By definition, sufficiently smooth $f$ vanishing near $\partial D$ belongs to $\mathcal{D}(\bar{A}_L)$ and $A_Lf=Af$.

In fact, for a functional $F(f)=(v,f)_s$ for $f\in\mathcal{K}$, there is $v^*$ such that $F(f)=(v^*,f)_l$ by Riesz theorem. Then, by Milgram-Lax theorem combined with (8), there is $u$ such that $F(f)=(v^*,f)_l=B_\alpha(u,f)$, which solves (11). Then, (12) extends the solvability of (11) for $v\in\mathcal{H}$, as in

**Lemma 2.** *For each $v\in\mathcal{H}$, (11) has a unique solution $u$ which satisfies (12). Thus, a bounded linear operator $G_\alpha=(\alpha-\overline{A}_L)^{-1}$ is defined on $\mathcal{H}$, and $\overline{A}_L$ is closed in $\mathcal{H}$.*

**Proposition 6.**

$$\alpha\|u\|_s^2+(\|u\|_l^2-\|u\|_s^2)=(u,v)_s, \qquad \text{for } u=G_\alpha v,\ v\in\mathcal{H}.$$

$$\|\alpha u-v\|_s^2+\alpha(\|u\|_l^2-\|u\|_s^2)=(u,v)_l-(u,v)_s, \qquad \text{for } u\in G_\alpha v,\ v\in\mathcal{K}.$$

$$\|\alpha u-v\|_l^2+(\alpha-1)\|\alpha u-v\|_s^2=(\alpha u-v,\overline{A}_L v)_s, \qquad \text{for } u=G_\alpha v,\ v\in\mathcal{D}(\overline{A}_L).$$

These are proved by using (10). By combing these equalities, we have

**Lemma 3.**

$$\alpha\|G_\alpha v\|_s\leqslant\|v\|_s,\ \lim_{\alpha\to\infty}\|\alpha G_\alpha v-v\|_s=0, \qquad \text{for } v\in\mathcal{H}$$

$$\alpha\|G_\alpha v\|_l\leqslant\|v\|_l,\ \lim_{\alpha\to\infty}\|\alpha G_\alpha v-v\|_l=0, \qquad \text{for } v\in\mathcal{K}.$$

**Theorem 2.** *$\overline{A}_L$ is the generator of a semigroup $\{T_t, t\geqslant 0\}$ on $\mathcal{H}$, which satisfies* (A. 2) *and* (A. 3). (A. 1), (A. 4) *and* (A. 5) *are satisfied for $\mathcal{H},\mathcal{K}$ and $\mathcal{A}=\overline{A}_L$. Hence, the group of operators $\{U_t, -\infty<t<\infty\}$ in Theorem* 1 *exists.*

In fact, (A. 2) and (A. 3) are proved by Lemmas 2 and 3 and Remark in 2 easily. (A. 4) is clear by (10) and the definition of $(f,f)_l$ with $\alpha_0=1$. (A. 5) is also clear by (10) and the symmetry of $B_\alpha(f,g)$: $(\overline{A}_L f,g)_s-(\overline{A}_L g,f)_s=B_\alpha(g,f)-B_\alpha(f,g)=0$.

Thus, the wave equation with Wentzell's boundary condition with $L$ in (2′) has been solved in the sense of the Corollary to Theorem 1.

**Note.** We assumed that $\tilde{A}u=\sum_1^{N-1}\alpha_{ij}(\partial^2 u/\partial\xi_i\partial\xi_j)+\sum_1^{N-1}\beta_i(\partial u/\partial\xi_i)+\gamma\cdot u$ in $Lu(x)$ is $a\cdot\sum_1^{N-1}(\partial^2 u/\partial\xi_i^2)$. But, if $\tilde{A}$ with smooth coefficients is uniformly elliptic or formally self adjoint, then (A. 1)–(A. 4) hold true. But, when $\tilde{A}$ is *not* formally self adjoint, our present proof needs $\delta(x)>0$ for (A. 5). For $\nu(x,\cdot)$, a much weaker condition is enough for (A. 1)–(A. 4). In fact, $\nu(x,\cdot)$ need not be concentrated on $\partial D$, or have a symmetric density. But, our present proof needs $\nu(x,D)<\infty$ for (A. 5). It is not clear that some technical device can remove this condition or not.

We can also construct $\{T_t\}$ and $\{U_t\}$ in a similar way to [3], by constructing a class of semigroups on $\partial D$ first. But, the present proof is much simpler.

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