178. A Generalization of the Closed Graph Theorem

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S. Banach ([3]) proved the closed graph theorem: every linear mapping with a closed graph from any complete metrizable topological vector space into another one is continuous, and also proved the open mapping theorem: every continuous linear mapping from any complete metrizable topological vector space onto another one is open. Many generalizations of these theorems have been tried. Recently, N. Adasch ([1] and [2]) and W. Robertson ([5]) introduced two classes of topological vector spaces: "infra-s-Raum" or "K_r-complete space" and "s-Raum" or "K-complete space" by N. Adasch or W. Robertson, respectivery. They proved that the former space is the (locally convex) Hausdorff topological vector space satisfying a minimal condition under which the closed graph theorem holds for all linear mappings from any ultrabarrelled (barrelled) spaces, and that the later space is the (locally convex) Hausdorff topological vector space satisfying a minimal condition under which the open mapping theorem holds for every linear mapping with a closed graph onto any ultrabarrelled (barrelled) space. N. Adasch ([2]) also proved that a Hausdorff topological vector space $F[\eta_0]$ is an infra-s-Raum if and only if η^{ut} coincides with η_0^{ut} for every Hausdorff vector topology η coarser than η_0, where $\eta^{ut}(\eta_0^{ut})$ is the ultrabarrelled topology associated with the topology $\eta(\eta_0)$. (Hereafter we assume that a notation $F[\eta]$ means the topological vector space F with the vector topology η.)

S. O. Iyahen ([4]) celled a topological vector space E **-inductive limit* of a family $\{E_\alpha\}$ of topological vector spaces by mappings $\{u_\alpha\}$, where for each α, u_α maps E_α into E and the union of the subspaces $u_\alpha(E_\alpha)$ spans E, if and only if the topology of E is the finest vector topology making every u_α continuous.

We shall generalize the Banach's theorems furthermore by these results. For locally convex spaces we have the same results, which are proved in this paper for general topological vector spaces, by using the concept of the inductive limit instead of the one of the *-inductive limit.

In this paper we consider that a property (p) on topological vector

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spaces satisfies the following conditions:

PI. every finite dimensional Hausdorff topological vector space has the property (p).

PII. every *-inductive limit of topological vector spaces with the property (p) has the property (p).

These conditions ensure that every topological vector space with the finest vector topology has the property (p); then we have next propositions.

Proposition 1. *For every topological vector space with a topology η, there exists the coarsest vector topology finer than η with which the space has the property (p).*

We denote this topology by η^p.

Proposition 2. *If a vector topology λ is finer than another one η, then λ^p is finer than η^p.*

Proposition 3. *For every continuous linear mapping from a topological vector space with the property (p) into a topological vector space with a vector topology η, it remains continuous when η is replaced by η^p.*

Proof. Let f be a continuous linear mapping from a topological vector space with the property (p) into a topological vector space $F[\eta]$. Let λ be the finest vector topology on F which makes f continuous, that is, λ is the *-inductive limit topology on F defined by f and all the injections of finite dimensional subspaces of F. Then λ is finer than η and, by Proposition 2, λ^p is finer than η^p. On the other hand, by Proposition 1, λ coincides with λ^p. Thus f is continuous into $F[\eta^p]$.

Proposition 4. *For every linear mapping with a closed graph from a topological vector space into a Hausdorff topological vector space, the *-inductive limit topology on the later space defined by the linear mapping and the identity mapping is Hausdorff.*

Proof. Let f be a linear mapping with a closed graph from a topological vector space E into a Hausdorff topological vector space F. Then the *-inductive limit topology defined by f and the identity mapping on F has a base of neighborhoods of 0 consisting of the sets $f(U)+V$, where U and V are neighborhoods of 0 in E and F, respectively. Thus the result is obtained by the closedness of the graph of f.

Definition 1. We call a Hausdorff topological vector space $F[\eta]$ K_r^p-*complete* if ζ^p coincides with η^p for every Hausdorff vector topology ζ coarser than η.

Theorem 1. *A Hausdorff topological vector space $F[\eta]$ is K_r^p-complete if and only if every linear mapping with a closed graph from any topological vector space with the property (p) into the space $F[\eta]$ is continuous.*

Proof. Suppose that $F[\eta]$ is K_r^p-complete. Let f be a linear mapping with a closed graph from a topological vector space with the property (p) into the space $F[\eta]$, and let ζ be the *-inductive limit topology on F defined by f and the identity mapping. Then ζ is Hausdorff by Proposition 4 and coarser than η. By Proposition 3, f is a continuous linear mapping into $F[\zeta^p]$. Since $F[\eta]$ is K_r^p-complete, ζ^p coincides with η^p. Therefore, by Proposition 1, f is a continuous linear mapping into $F[\eta]$.

Conversely, suppose ζ is a Hausdorff vector topology coarser than η. By Proposition 2 ζ^p is coarser than η^p. Let I be the identity mapping from $F[\zeta^p]$ onto $F[\eta]$. Then, ζ being coarser than both of ζ^p and η, the graph of I is closed, and by the assumption, I is continuous. Thus ζ^p is finer than η and so than η^p. Therefore ζ^p coincides with η^p and $F[\eta]$ is K_r^p-complete.

Corollary 1. *A Hausdorff topological vector space $F[\eta]$ is K_r^p-complete if and only if every one-to-one linear mapping with a closed graph from any topological vector space with the property* (p) *onto $F[\eta]$ is continuous.*

Corollary 2. *Every one-to-one linear mapping with a closed graph from a K_r^p-complete space onto a topological vector space with the property* (p) *is open.*

Corollary 3. *A closed subspace and a continuous linear one-to-one image of a K_r^p-complete space are also K_r^p-complete.*

Definition 2. We call a topological vector space $F[\eta]$ *K^p-complete* if every Hausdorff quotient space of F is K_r^p-complete.

Clearly each Hausdorff K^p-complete space is K_r^p-complete.

Theorem 2. *A topological vector space $F[\eta]$ is K^p-complete if and only if every linear mapping with a closed graph from $F[\eta]$ onto any topological vector space with the property* (p) *is open.*

Proof. Suppose $F[\eta]$ is a K^p-complete space and f is a linear mapping with a closed graph from $F[\eta]$ onto a topological vector space E with the property (p). Then there exists a linear one-to-one mapping u from $F/f^{-1}(0)$ onto E such that $f=u\cdot q$, where q is the quotient mapping from F onto $F/f^{-1}(0)$. As the graph of f is closed, $f^{-1}(0)$ is a closed subspace of F and u has a closed graph. Then u^{-1} is a linear mapping from E onto the K_r^p-complete space $F/f^{-1}(0)$ with a closed graph, and it is continuous by Theorem 1. Therefore u is an open mapping and so is f.

Conversely suppose that G is a closed subspace of F, and s is a linear one-to-one mapping with a closed graph of a topological vector space E with the property (p) onto F/G. Then, for the quotient mapping q from F onto F/G, the mapping $s^{-1}\cdot q$ from F onto E has a closed graph and it is an open mapping by the assumption. Hence, by Corollary 1 of Theorem 1, F/G is K_r^p-complete. Therefore $F[\eta]$ is a K^p-complete space.

Corollary. *A closed subspace and a continuous linear image of a K^p-complete space are also K^p-complete.*

As a property (p), we can take, for example, "ultrabarrelled", "ultrabornological" or "quasi-ultrabarrelled", since these properties satisfy the conditions PI and PII (S. O. Iyahen [4]). In the case of "ultrabarrelled", a K_r^p-complete [K^p-complete] space is the same as an infra-s-Raum [s-Raum] defined by N. Adasch or a K_r-complete space [K-complete space, respectively] by W. Robertson.

When $F[\eta]$ is a Hausdorff topological vector space with a property (p), $F[\eta]$ is K_r^p-complete if and only if η is a minimal vector topology on F which is Hausdorff and has the property (p). If we take nothing for a property (p), that is, we only consider the topological vector space in this case, then we have the following theorem by Theorem 1.

Theorem 3. *A topology η of a topological vector space F is minimal Hausdorff if and only if every linear mapping with a closed graph from any topological vector space into $F[\eta]$ is continuous.*

Added in proof. In the case of locally convex spaces, M. H. Powell [5] obtained the same results more concretely.

References

[1] N. Adasch: Tonnelierte Räume und zwei Sätze von Banach. Math. Ann., **186**, 209–214 (1970).

[2] ——: Der Graphensatz in topologischen Vektorräumen. Math. Z., **119**, 131–142 (1971).

[3] S. Banach: Théorie de opérations linéaires. Warszawa (1932).

[4] S. O. Iyahen: On certain classes of linear topological spaces. Proc. London Math. Soc., **18**(3), 285–307 (1968).

[5] M. H. Powell: On Kōmura's closed-graph theorem. Technical Report TR 72-35, Department of Mathematics, University of Maryland (1972).

[6] W. Robertson: On the closed graph theorem and spaces with webs. Proc. London Math. Soc., **24**(3), 692–738 (1972).