177. *On a Decomposition of Automorphisms of von Neumann Algebras*

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1. Recently, in [1], Borchers has characterized inner automorphisms of von Neumann algebras. In the paper, he also investigated and classified general automorphisms of von Neumann algebras. As an interesting consequence of Theorem 3.8 and Theorem 4.1 in [1], we have the following theorem (cf. Remark in below).

Theorem. *Let $\mathcal{A}$ be a von Neumann algebra, $\mathcal{Z}$ the center of $\mathcal{A}$, α an automorphism of $\mathcal{A}$ and $\mathcal{B}$ the fixed algebra of α, that is, $\mathcal{B}=\{A \in \mathcal{A}; \alpha(A)=A\}$. Then there exists a sequence of mutually orthogonal projections $\{E_n; n=0,1,2,\cdots\}$ in $\mathcal{Z} \cap \mathcal{B}$ which satisfies the following conditions:*

(1) $\sum_{n=0}^{\infty} E_n = I$,

(2) *for each $n \neq 0$, α^k is inner on $\mathcal{A}_{E_n}$ for $k \equiv 0 \pmod n$,*

(3) *for each $n \neq 0$, α^k is freely acting on $\mathcal{A}_{E_n}$ for $k \not\equiv 0 \pmod n$, and*

(4) *α^k is freely acting on $\mathcal{A}_{E_0}$ for $k=1,2,\cdots$.*

In this paper, we shall show, without applying Theorem 3.8 and Theorem 4.1 in [1], the Theorem using the Kallman decomposition theorem of automorphisms [4: Theorem 1.11].

2. Let $\mathcal{A}$ be a von Neumann algebra and α an automorphism of $\mathcal{A}$ (by an automorphism of a von Neumann algebra we mean an automorphism for the $*$-algebra structure). α is called *freely acting* on $\mathcal{A}$ if

$$AB=\alpha(B)A \qquad \text{for any } B \in \mathcal{A}$$

implies $A=0$ ([4]). If F is a projection in the center of $\mathcal{A}$ fixed by α, we can consider α an automorphism of the reduced von Neumann algebra $\mathcal{A}_F$ of $\mathcal{A}$ by the equality

$$\alpha(AF)=\alpha(A)F \qquad \text{for any } A \in \mathcal{A}.$$

By Kallman's theorem, there exists a central projection F fixed under α such that α is inner on $\mathcal{A}_F$ and α is freely acting on $\mathcal{A}_{I-F}$. We shall call this projection F the *central projection inducing the inner part* of α.

Remark. By Kallman's theorem, we have that α is freely acting on $\mathcal{A}$ if and only if α is outer on $\mathcal{A}_G$ for each central projection G fixed under α. Hence, our Theorem is an immediate result of Theorem

3.8 and Theorem 4.1 in [1].

In order to prove the Theorem by using the Kallman decomposition theorem we need the following Lemma, which is a variation of Lemma 4 in [3] (or, cf. [2: Lemma 1]) and is proved with a minor modification.

Lemma. *Let $\mathcal{A}$ be a von Neumann algebra, G a group of automorphisms of $\mathcal{A}$ and α an automorphism of $\mathcal{A}$ such that*

$$\alpha g = g\alpha$$

for every $g \in G$. Then the central projection F inducing the inner part of α is fixed under every $g \in G$, that is,

$$g(F) = F \qquad \text{for every } g \in G.$$

3. Now, we shall prove the Theorem. Let E_1 be the central projection inducing the inner part of α, then $E_1 \in \mathcal{B}$. Consider α an automorphism of $\mathcal{A}_{I-E_1}$. Let E_2 be the central projection inducing the inner part of α^2 in $\mathcal{A}_{I-E_1}$, then E_2 is a central projection of $\mathcal{A}$ such as $E_2 \leqq I - E_1$. Applying Lemma to α^2 and $G = \{\alpha^n ; n = 0, \pm 1, \pm 2, \cdots\}$, we have $E_2 \in \mathcal{B}$. Therefore, there exists a projection $E_2 \in \mathcal{Z} \cap \mathcal{B}$ such that α is freely acting on $\mathcal{A}_{E_2}$, α^2 is inner on $\mathcal{A}_{E_2}$ and α^2 is freely acting on $(\mathcal{A}_{I-E_1})_{I-E_2} = \mathcal{A}_{I-E_1-E_2}$. Repeating this method inductively, for every positive integer n, there exists a projection $E_n \in \mathcal{Z} \cap \mathcal{B}$ which satisfies the following three conditions:

$E_n \leqq I - (E_1 + \cdots + E_{n-1})$,

α^n is inner on $\mathcal{A}_{E_n}$, and

α^n is freely acting on $\mathcal{A}_{I-(E_1+\cdots+E_n)}$.

Put $E_0 = I - \sum_{n=1}^{\infty} E_n$, then it is clear that $E_n (n = 0, 1, 2, \cdots)$ are projections in $\mathcal{Z} \cap \mathcal{B}$ satisfying the conditions (1) and (2). For two positive integers k and n, assume that $k \not\equiv 0 \pmod{n}$. So, k has a form $k = ln + m$ $(l = 0, 1, 2, \cdots ; m = 1, 2, \cdots, n-1)$. Then α^{ln} is inner on $\mathcal{A}_{E_n}$ and α^m is freely acting on $\mathcal{A}_{E_n}$ because $E_n \leqq I - (E_1 + E_2 + \cdots + E_m)$, and so $\alpha^k = \alpha^{ln} \cdot \alpha^m$ is freely acting on $\mathcal{A}_{E_n}$. Thus E_n $(n = 1, 2, \cdots)$ satisfy the condition (3). By the definition, for every integer k, $E_0 \leqq I - (E_1 + \cdots + E_k)$. It implies that α^k is freely acting on $\mathcal{A}_{E_0}$, which is the condition (4). Thus Theorem is proved.

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References

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