176. *Operator Norms as Bounds for Roots of Algebraic Equations*

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1. Introduction. Very recently, Ifantis and Kouris [1] show, a Hilbert space approach is powerful to give bounds of roots of algebraic equations; actually, they show that the operator bound of a perturbation of the simple unilateral shift by a dyad gives certain bounds of roots. In the present note, giving three norms on n-dimensional vector space, we shall obtain certain bounds of roots estimating operator norms of companion matrices.

For a given algebraic equation

$$p(z)=z^n+a_nz^{n-1}+\cdots+a_1=0, \tag{1}$$

we associate the *companion matrix*

$$T=\begin{pmatrix} -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_2 & -a_1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ & \cdots & & \cdots & \cdots & \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \tag{2}$$

cf. [2], esp. Chapter VII. It is well-known that *the spectrum* $\sigma(T)$ *of* T *coincides with the set of all roots of* (1), i.e.

$$\sigma(T)=\{z\,;\,p(z)=0\}. \tag{3}$$

From (3), we have

$$|z|\leqq r(T)\leqq\|T\|, \tag{4}$$

for any root z of (1), where $r(T)$ is the spectral radius of T: $r(T)=\sup_{z\in\sigma(T)}|z|$ and $\|T\|$ is the operator norm of T: $\|T\|=\sup_{\|f\|=1}\|Tf\|$ considering T as an operator on the n-dimensional Banach space H.

2. Carmichael-Mason's theorem. Here we regard H as the n-dimensional unitary space with orthonormal basis $e_1,\cdots,e_n$. For x, $y\in H$, we put $(x\otimes y)z=(z,y)x$ for $z\in H$. Then we can express the companion matrix T of (1) as

$$T=V-e_1\otimes u, \tag{5}$$

where

$$u=a_n^*e_1+\cdots+a_1^*e_n \tag{6}$$

and

$$V=\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ & \cdots & & \cdots & & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}. \tag{7}$$

Theorem 1. *If z is a root of* (1), *then*

$$|z|\leqq 1+|a_1|+\cdots+|a_n|. \tag{8}$$

Proof. We have

$$\begin{aligned}\|T\|&=\|V-e_1\otimes u\|=\|V-\sum_{i=1}^{n} a_{n-i+1}e_1\otimes e_i\|\\ &\leqq\|V\|+\sum_{i=1}^{n}|a_{n-i+1}|\,\|e_1\|\,\|e_i\|\\ &\leqq 1+\sum_{i=1}^{n}|a_i|.\end{aligned}$$

Hence we have (8) by (4).

Theorem 2 (Carmichael-Mason). *If z is a root of* (1), *then*

$$|z|\leqq(1+|a_1|^2+\cdots+|a_n|^2)^{1/2}. \tag{9}$$

Proof. Since $V^*e_1=0$, we have $V^*(e_1\otimes u)=V^*e_1\otimes u=0$. Hence we have

$$\begin{aligned}\|T\|^2&=\|T^*T\|=\|(V-e_1\otimes u)^*(V-e_1\otimes u)\|\\ &=\|V^*V+u\otimes u\|\leqq\|V^*V\|+\|u\otimes u\|\\ &\leqq 1+\|u\|^2.\end{aligned}$$

Since $\|u\|^2=|a_1|^2+\cdots+|a_n|^2$, we have (9) by (4).

3. Montel's and Eneström-Kakeya's Theorem. We shall replace the norm of H by the sup-norm:

$$\|f\|_\infty=\max\{|f_1|,\cdots,|f_n|\} \tag{10}$$

for $f=(f_1,\cdots,f_n)\in H$.

Theorem 3 (Montel). *If z is a root of* (1), *then*

$$|z|\leqq\max\{1,|a_1|+\cdots+|a_n|\}. \tag{11}$$

Proof. For a matrix $X=(x_{ij})$, we have

$$\|X\|_\infty=\max_{1\leqq i\leqq n}\sum_{j=1}^{n}|x_{ij}|.$$

Therefore, we have (11) by (4).

Theorem 4 (Montel). *If z is a root of* (1), *then*

$$|z|\leqq|a_1|+|a_1-a_2|+\cdots+|a_{n-1}-a_n|+|a_n-1|. \tag{12}$$

Proof. Put $q(z)=(1-z)p(z)$. Apply Theorem 3 for $q(z)$. Since the right hand side of (12) is not less than 1, we have (12) by (11).

It is known that Theorem 4 implies the following well-known theorem:

Theorem 5 (Eneström-Kakeya). *If z is a root of*

$$b_0z^n+b_1z^{n-1}+\cdots+b_n=0,$$

where

$$b_0\geqq b_1\geqq\cdots\geqq b_n\geqq 0,$$

then $|z|\leqq 1$.

4. Cauchy's theorem. If we replace the norm of B by

$$\|f\|_1=|f_1|+\cdots+|f_n|, \tag{13}$$

for $f=(f_1,\cdots,f_n)$, then we have another bound for roots of (1):

Theorem 6 (Cauchy). *If z is a root of* (1), *then*

$$|z|\leqq 1+\max\{|a_1|,\cdots,|a_n|\}. \tag{14}$$

Proof. It is sufficient to show that $\|T\|_1$ is not greater than the right hand side of (14). For any $f\in H$, we have

$$\begin{aligned}\|Tf\|_1&=\sum_{i=1}^{n}|a_{n-i+1}f_i|+\sum_{i=1}^{n-1}|f_i|\\&\leqq\Big(1+\max_{1\leqq i\leqq n}|a_i|\Big)\|f\|_1.\end{aligned}$$

5. Operator coefficients. In the preceding sections, we have calculated bounds for roots of equations with numerical coefficients. In this section, we shall give bounds for roots of equations with operator coefficients. Similar equations are considered, as a generalization of the classical propervalue problem, by Atkinson, Sz.-Nagy, Müller and others, cf. [3].

Let us suppose that

$$V(z)=z^n+z^{n-1}V_n+\cdots+zV_2+V_1$$

where $V_1,\cdots,V_n$ are (bounded linear) operators on a Hilbert space K. The (operator) companion matrix of $V(z)$ is

$$V=\begin{pmatrix}-V_n & -V_{n-1} & -V_{n-2} & \cdots & -V_2 & -V_1\\ I & 0 & 0 & \cdots & 0 & 0\\ 0 & I & 0 & \cdots & 0 & 0\\ & \cdots & & \cdots & \cdots & \\ 0 & 0 & 0 & \cdots & I & 0\end{pmatrix}.$$

Let H be the direct sum of n copies of K. We shall consider V as a linear operator on H. A complex number z is called a *root* of $V(z)$ if there is a non-zero $x\in K$ such that

$$V(z)x=0. \tag{15}$$

As in the case of numerical coefficients, we have

Lemma 7. *The set of all roots of* (15) *is the point spectrum $\sigma_P(V)$ of V.*

Proof. If $z\in\sigma_P(V)$, then $Vx=zx$ for some non-zero $x=(x_1,\cdots,x_n)\in H$. Hence we have

$$\begin{array}{llllll}-V_nx_1-V_{n-1}x_2-\cdots-V_2x_{n-1}-V_1x_n&=zx_1\\ x_1&=zx_2\\ \quad x_2&=zx_3\\ \qquad\cdots\quad\cdots&\cdots\\ \qquad\qquad x_{n-1}&=zx_n.\end{array} \tag{16}$$

From (16), we have $x_n\neq 0$ and $V(z)x_n=0$, that is, z is a root of (15).

Conversely, if $V(z)x_0=0$ for $x_0\neq 0$, then we have a non-zero vector

$x \in K$ by (16) putting $x_n = x_0$ and $x = (x_1, \cdots, x_n)$. Clearly, x satisfies $Vx = zx$, and $z \in \sigma_P(V)$.

By Lemma 7, corresponding to (4), we have

$$|z| \leqq r(V) \leqq \|V\|, \tag{4'}$$

for any root z of (15). Hence, we can give a bound for roots of (15) estimating $\|V\|$.

For example, if we give a norm of H by

$$\|f\|_\infty = \max\{\|f_1\|, \cdots, \|f_n\|\} \quad \text{for} \quad f = f_1 \oplus \cdots \oplus f_n, \tag{17}$$

then we have following theorem corresponding to Montel's theorem:

Theorem 8. *If z is a root of* (15), *then*

$$|z| \leqq \max\{1, \|V_1\| + \cdots + \|V_n\|\}. \tag{18}$$

Proof. For any $x \in H$ such that $\|x\|_\infty \leqq 1$, we have

$$\begin{aligned} \|Vx\|_\infty &\leqq \max\left\{\sum_{i=1}^{n} \|V_{n-i+1}\| \cdot \|x_i\|, \|x_1\|, \cdots, \|x_{n-1}\|\right\} \\ &\leqq \max\{1, \|V_1\| + \cdots + \|V_n\|\}. \end{aligned}$$

Hence, by (4′), we have (18).

Corresponding to Theorem 4, we have

Theorem 9. *If z is a root of* (15), *then*

$$|z| \leqq \|V_1\| + \|V_1 - V_2\| + \cdots + \|V_{n-1} - V_n\| + \|V_n - I\|. \tag{19}$$

References

[1] E. K. Ifantis and C. B. Kouris: A Hilbert space approach to the localization problem of the roots of analytic functions. Indiana Univ. Math. J., **23**, 11–22 (1973).

[2] M. Marden: The Geometry of the Zeros of a Polynomial in a Complex Variable. Math. Survey, No. 3, Amer. Math. Soc., Providence (2nd. ed.) (1960).

[3] P. H. Müller: Zu einer Spektralbetrachtung von Atkinson und Sz.-Nagy. Acta Sci. Math. Szeged, **17**, 195–197 (1956).