175. *Index Theorem for a Maximally Overdetermined System of Linear Differential Equations*

By Masaki KASHIWARA
Research Institute for Mathematical Sciences, Kyoto University

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In this note we state the index theorem for a maximally overdetermined system of linear partial differential equations. The theorem comprises as a special case the already known index theorem for an ordinary differential equation (Kashiwara [2], Komatsu [4] and Malgrange [5]).

1. Local characteristic. Let (S, x) be a germ of an irreducible analytic space. We define the local characteristic $c_x(S)$ by the induction on the dimension of S as follows.

We embed (S, x) into an enclidean space $(C^N, 0)$ and choose a Whitney stratification $S=\cup S_\alpha$ of S. The open stratum of S is denoted by S_0. Let d_α be the dimension of S_α and x_α be a point in S_α. We define $c_x(S)$ inductively by the following formula

$$c_x(S)=\sum_{S_\alpha \neq S_0} c_x(\bar{S}_\alpha)\chi(U_\alpha \cap S_0 \cap Z_\alpha)$$

where U_α denotes a sufficiently small open ball with center x_α, Z_α denotes a $(d_\alpha+1)$-codimensional linear variety in a generic position in C^N sufficiently close to x_α, χ denotes the Euler characteristic and the sum extends over all the strata S_α other than S_0.

Proposition. *The definition of a local characteristic $c_x(S)$ is independent of the choice of the embedding $(S, x)\subset(C^N, 0)$ and the stratification.*

We will give the expamples of local characteristics.

Example 1. If (S, x) is non singular, then $c_x(S)=1$.

Example 2. If (S, x) is a hypersurface in C^{n+1} with the isolated singularity at x, then $c_x(S)=1+(-1)^{n-1}\mu$ where μ is the Milnor number of the generic hyperplane section of S through the point x. In particular, for $S=\{x\in C^{n+1}; x_0^{p_0}+\cdots+x_n^{p_n}=0\}$, we have $c_0(S)=1+(-1)^{n-1}(p_1-1)\cdots(p_n-1)$ with $p_0=\max_j p_j$

Example 3. If (S, x) is a curve, then $c_x(S)$ coincides with the multiplicity of S at x.

Example 4. If $S=\{(x, y, z)\in C^3; x^n+y^pz^q=0\}$ (g. c. d. $(p, q, n)=1$ and $p, q, n\geqq 1$), then $c_0(S)=\min(n, p)+\min(n, q)-n$.

2. Index theorem. Let X be a complex manifold, $\mathcal{O}$ be the sheaf of holomorphic functions on X, $\mathcal{D}$ be the sheaf of differential operators of finite order on X and $\mathcal{M}$ be a system of differential equations, that is, a coherent left $\mathcal{D}$-Module (as for the notations we refer the reader to Sato-Kawai-Kashiwara [1], Kashiwara [2], [3]).

Let $\mathcal{M}=\cup\mathcal{M}_k$ be a good filtration of $\mathcal{M}$, namely, a filtration by such coherent $\mathcal{O}$-Modules $\mathcal{M}_k$ that one has $\mathcal{D}_l\mathcal{M}_k\subset\mathcal{M}_{k+l}$ for any l, k and $\mathcal{D}_l\mathcal{M}_k=\mathcal{M}_{l+k}$ for $k\gg 0$ with $\mathcal{D}_l$ denoting the sheaf of differential operators of order $\leqq l$. $\widehat{SS}(\mathcal{M})$ is the support of the coherent sheaf $\widetilde{gr\mathcal{M}}$ on T^*X associated with $gr\mathcal{M}=\oplus(\mathcal{M}_k/\mathcal{M}_{k-1})$. For an irreducible analytic subset Λ in T^*X, the multiplicity of $\mathcal{M}$ at Λ is by the definition the multiplicity of $\widetilde{gr\mathcal{M}}$ at Λ. These notions are independent of the choice of the good filtration.

We assume that $\mathcal{M}$ is maximally overdetermined. This means the dimension of $\widehat{SS}(\mathcal{M})$ equals to that of X.

By definition the index $\chi_x(\mathcal{M})$ of $\mathcal{M}$ at a point x of X is given by

$$\chi_x(\mathcal{M})=\sum(-1)^i \dim_C \mathcal{E}xt^i_{\mathcal{D}}(\mathcal{M},\mathcal{O})_x.$$

$\widehat{SS}(\mathcal{M})$ is expressed as union

$$\widehat{SS}(\mathcal{M})=\bigcup_j \overline{T^*_{Y'_j}X}$$

in a neighborhood of x where Y'_j is a non singular locus of an analytic subset Y_j of X irreducible at x. This expression is unique.

Theorem. $\chi_x(\mathcal{M})=\sum_j(-1)^{d_j}c_x(Y_j)m_j$
where d_j is the codimension of Y_j and m_j is the multiplicity of $\mathcal{M}$ at $T^*_{Y'_j}X$.

This theorem is derived from the structure theorem for a maximally overdetermined system of pseudo-differential equations and the study of the sheaf $\mathcal{E}xt^j_{\mathcal{D}}(\mathcal{M},\mathcal{O})$ (see [3]).

References

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