174. *On the Structure of Certain Types of Polarized Varieties*

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1. This is a report on our recent results on a study of structures of polarized varieties. Details will be published elsewhere.

In this note we mean by an algebraic variety a complex space associated with an irreducible, reduced and proper $\boldsymbol{C}$-scheme. We fix our notation.

$c_j(E)$: the j-th Chern class of a vector bundle E,
$\boldsymbol{P}(E)$: the projective bundle associated with E,
$\boldsymbol{L}(E)$: the tautological line bundle on $\boldsymbol{P}(E)$,
E^*: the dual vector bundle of E,
$|F|$: the complete linear system of Cartier divisors associated with a line bundle F,
B_sL: the set of base points of a linear system L,
$[W]$: the natural integral base of $H_{2n}(W;\boldsymbol{Z})$ where W is a variety of dimension n,
K_M: the canonical line bundle on a manifold M.

Let F be an ample line bundle on a variety V. We call such a pair (V,F) a polarized variety. In addition if V is non-singular we call (V,F) a polarized manifold. We say that (V_1,F_1) is isomorphic to (V_2,F_2) and write $(V_1,F_1)\cong(V_2,F_2)$ if there is a biholomorphic mapping $f:V_1\to V_2$ such that $F_1=f^*F_2$. We define the following invariants of a polarized variety (V,F) of dimension n:

$d(V,F)=F^n=(c_1(F))^n[V]$,
$\Delta(V,F)=\dim V+d(V,F)-\dim H^0(V,\mathcal{O}_V(F))$,
and if V is non-singular, we define
$g(V,F)=(K_V+(n-1)F)F^{n-1}/2+1$.

The importance of $\Delta(V,F)$ is illustrated by the following fact.

Lemma A. *Let (V,F) be a polarized variety. Then $\dim B_s|F| <\Delta(V,F)$, where $\dim\emptyset$ is defined to be -1. In particular $\Delta(V,F)\geqq 0$ for every polarized variety.*

In section 2 we give a complete classification of polarized manifolds with $\Delta=0$. In section 3 we give certain structure theorems concerning polarized manifolds with $\Delta=1$, and classify such manifolds except the case in which $d=5,6$ and $\dim M=3$.

Our proof by induction with respect to the dimension of the variety V is based on a detailed analysis of the rational mappings associated with certain linear systems on V. Hironaka's theorems on resolution of singularities are essential to our method.

2. The case in which $\Delta=0$.

Theorem B. *Let (V,F) be a polarized variety with $d(V,F)=1$. Then the following conditions are equivalent to each other:*

a) *V is non-singular and $g(V,F)=0$,*

b) *$\Delta(V,F)=0$,*

c) *$(V,F)\cong(\boldsymbol{P}^n,H)$ where n is the dimension of V and H is the hyperplane bundle on the projective space $\boldsymbol{P}^n$.*

Theorem C. *Let (V,F) be a polarized variety with $d(V,F)=2$. Then the following conditions are equivalent to each other:*

a) *$\Delta(V,F)=0$,*

b) *V is isomorphic to a quadric in $\boldsymbol{P}^{n+1}$ and F is the hyperplane bundle on it.*

If V is non-singular, each of a) *and* b) *is equivalent to the following:*

c) *$g(V,F)=0$.*

The idea of proofs of these theorems can be found in Kobayashi-Ochiai [3].

Theorem D. *Let (M,F) be a polarized manifold with $\Delta(M,F)=0$, $d(M,F)\geqq 3$ and $\dim M=n$. Then, except the case in which $(M,F)\cong(\boldsymbol{P}^2,2H)$ there exists a vector bundle E on $\boldsymbol{P}^1$ which is a direct sum of ample line bundles such that $(M,F)=(\boldsymbol{P}(E^*),-\boldsymbol{L}(E^*))$. The linear system $|K_M+nF|$ gives the bundle mapping.*

3. The case in which $\Delta=1$.

Lemma E. *Let (M,F) be a polarized manifold with $\Delta(M,F)=1$. Then a general member of $|F|$ is non-singular.*

Proposition F. 1) *Let (M,F) be a polarized manifold with $\Delta(M,F)=1$, $d(M,F)=1$ and $\dim M=n$. Then $B_s|F|$ contains only one point p. Moreover, if $\pi:\tilde{M}\to M$ is the monoidal transformation of M with center p, then $B_s|\pi^*F-E_p|=0$ where $E_p=\pi^{-1}(p)$. The holomorphic mapping $\Phi:\tilde{M}\to\boldsymbol{P}^{n-1}$ associated with $|\pi^*F-E_p|$ satisfies the following conditions:*

a) *Φ is surjective,*

b) *E_p is a global section of Φ,*

c) *every fiber of Φ is an irreducible curve,*

d) *the genus of a general fiber is positive, and is equal to $g(M,F)$.*

2) *Suppose, conversely, that a holomorphic mapping $\Psi:\tilde{N}\to\boldsymbol{P}^{n-1}$ satisfying the above conditions* a), c), d) *are given, and that there exists a holomorphic section E of Ψ such that the normal bundle of it is the dual of the hyperplane bundle. Then there exists a polarized manifold (N,F) with $\Delta=d=1$ such that the holomorphic mapping*

obtained by the method described in 1) *of this theorem is equivalent to the given mapping* Ψ.

Lemma G. *Let* (M,F) *be a polarized manifold with* $\Delta(M,F)=1$ *and* $d(M,F)\geqq 2$. *Then* $B_s|F|=\emptyset$.

Theorem H. *Let* (M,F) *be a polarized manifold with* $\Delta(M,F)=1$, $d(M,F)=2$ *and* $\dim M=n$. *Then* M *is a two-sheeted branched covering manifold of* $\boldsymbol{P}^n$ *with a non-singular branch locus of degree* $2g(M,F)+2\geqq 4$, *and* F *is the pull back of the hyperplane bundle.*

Proposition I. *Let* (M,F) *be a polarized manifold of dimension* n. *Then* $\Delta(M,F)=g(M,F)=1$ *if and only if* K_M *is linearly equivalent to* $-(n-1)F$.

Corollary J. *Let* (M,F) *be a polarized manifold with* $\Delta(M,F)=1$, $g(M,F)=1$ *and* $\dim M=2$. *Then* M *is obtained by blowing up certain points in* $\boldsymbol{P}^2$ *except the case in which* $M\cong\boldsymbol{P}^1\times\boldsymbol{P}^1$.

Theorem K. *Let* (M,F) *be a polarized manifold with* $\Delta(M,F)=1$, $d(M,F)\geqq 3$. *Then* $g(M,F)=1$ *and* F *is very ample.*

Corollary K-1. *Let* (M,F) *be the same as in Theorem K and suppose that* $d(M,F)=3$ *and* $\dim M=n$. *Then* M *is isomorphic to a non-singular cubic in* $\boldsymbol{P}^{n+1}$ *and* F *is the hyperplane bundle on it.*

Corollary K-2. *Let* (M,F) *be the same as in Theorem K, and suppose that* $d(M,F)=4$ *and* $\dim M=n$. *Then* M *is isomorphic to a non-singular complete intersection of type* $(2,2)$ *in* $\boldsymbol{P}^{n+2}$ *and* F *is the hyperplane bundle on it.*

Theorem L. *Let* (M,F) *be a polarized manifold with* $\Delta(M,F)=1$, $d(M,F)\geqq 7$ *and* $\dim M\geqq 3$. *Then* $d(M,F)=7$ *or* 8. *Moreover,*
$(M,F)\cong(\boldsymbol{P}^3,2H)$ *when* $d(M,F)=8$, *and*
$(M,F)\cong(Q_p(\boldsymbol{P}^3),2H-E_p)$ *when* $d(M,F)=7$, *where* $Q_p(\boldsymbol{P}^3)$ *denotes the monoidal transformation of* $\boldsymbol{P}^3$ *with center* $p\in\boldsymbol{P}^3$, E_p *and* H *denote respectively the exceptional divisor and the hyperplane bundle on* $\boldsymbol{P}^3$.
The Theorem 4-1-1 of Deligne [1] plays an important role in our proof of Theorem L.

Remark. There are some examples of polarized manifolds with $\Delta(M,F)=1$, $d(M,F)=5,6$ and $\dim M\geqq 3$. But it is expected that there are very few types of such polarized manifolds.

References

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