170. *Maximum Principles for Implicit Parabolic Equations*

By Norio YOSHIDA
Department of Mathematics, Hiroshima University

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In [6], Nirenberg derived a strong maximum principle for second order linear parabolic equations. This result was extended by Besala [2] to nonlinear parabolic equations of the form

$$(*)\qquad u_t=f(t, x, u, u_x, u_{xx}),$$

where $x=(x_1,\cdots,x_n)$, $u_t=\partial u/\partial t$, $u_x=(\partial u/\partial x_i)_{i=1}^n$ and $u_{xx}=(\partial^2 u/\partial x_i\partial x_j)_{i,j=1}^n$.

On the other hand, Picone [7] and Krzyżański [4] established a maximum principle in unbounded domains which was particularly useful to the study of the Cauchy problem for second order linear parabolic equations. An extension of this principle to nonlinear equations of the form $(*)$ was given by Besala [1].

The purpose of this paper is to generalize the above mentioned results of Besala to the implicit parabolic equation

$$(1)\qquad F(t, x, u, u_t, u_x, u_{xx})=0.$$

Let D be a domain in the $(n+1)$-dimensional Euclidean space R^{n+1} of points (t, x). For each fixed point $(t^0, x^0)\in D$ we define $S_D^+(t^0, x^0)[S_D^-(t^0, x^0)]$ to be the set of all points $(t, x)\in D$ which can be joined to (t^0, x^0) by a upward [downward] directed broken line contained in D, with (t^0, x^0) as initial point and (t, x) as endpoint.

Consider a function $F(t, x, z, p, Q, R)$ defined for all $(t, x)\in D, z, p$, $Q=(q_i)_{i=1}^n$ and $R=(r_{ij})_{i,j=1}^n$. The function $F(t, x, z, p, Q, R)$ is said to belong to the class $\mathscr{P}(D)$ if there exist positive constants κ and τ such that

$$(2)\qquad F(t, x, z, p, Q, R)-F(t, x, z, \tilde{p}, Q, \tilde{R})\geqq\kappa\sum_{i=1}^n(r_{ii}-\tilde{r}_{ii})+\tau(\tilde{p}-p)$$

for all $(t, x)\in D, z, p, \tilde{p}$ with $p\leqq\tilde{p}$, Q, and symmetric matrices $R=(r_{ij})$, $\tilde{R}=(\tilde{r}_{ij})$ such that $R-\tilde{R}$ is positive semidefinite.

First, we shall prove a strong maximum principle for equation (1) which extends a recent result of Besala [2].

Theorem 1. *Assume that the function $F(t, x, z, p, Q, R)$ belongs to the class $\mathscr{P}(K)$ for any compact subset K of D, and that there exist positive constants L_0, L_1, L_2 and L_3 such that*

$$(3)\qquad \begin{aligned}&|F(t, x, z, p, Q, R)-F(t, x, \tilde{z}, \tilde{p}, \tilde{Q}, \tilde{R})|\\ &\leqq L_0|z-\tilde{z}|+L_1|p-\tilde{p}|+L_2\sum_{i=1}^n|q_i-\tilde{q}_i|+L_3\sum_{i,j=1}^n|r_{ij}-\tilde{r}_{ij}|\end{aligned}$$

for all $(t, x)\in K, z, \tilde{z}, p, \tilde{p}, Q, \tilde{Q}, R$ and $\tilde{R}$.

Let $u(t, x)$ and $v(t, x)$ be continuous and continuously differentiable *in D once with respect to t and twice with respect to x, and satisfy there the following inequalities:*

(4) $$u(t,x)\leqq v(t,x),$$

(5) $$F(t,x,v,v_t,v_x,v_{xx})\leqq F(t,x,u,u_t,u_x,u_{xx}).$$

If there exists a point $(t^0,x^0)\in D$ *such that* $u(t^0,x^0)=v(t^0,x^0)$, *then*

(6) $$u(t,x)\equiv v(t,x) \qquad in\ S_D^-(t^0,x^0).$$

Proof. The method is an adaptation of that used by Besala [2]. Suppose that (6) does not hold, and let (t^1,x^1) be a point in $S_D^-(t^0,x^0)$ at which $u(t^1,x^1)<v(t^1,x^1)$. Let σ be a downward directed broken line joining (t^0,x^0) to (t^1,x^1). There exists a point $(\bar t,\bar x)\in\sigma$ such that $u(\bar t,\bar x)=v(\bar t,\bar x)$ and $u(t,x)<v(t,x)$ at all points lying on σ between $(\bar t,\bar x)$ and (t^1,x^1). Thus in a neighborhood of $(\bar t,\bar x)$ contained in $S_D^-(t^0,x^0)$ there is a point $(\tilde t,\tilde x)$ with $\tilde t<\bar t$ at which $u(\tilde t,\tilde x)<v(\tilde t,\tilde x)$. We define the functions

$$\psi(t,x)=\rho^2-|x-\tilde x-\xi(t-\tilde t)|^2, \qquad \phi(t,x)=\psi^2(t,x)e^{-\alpha(t-\tilde t)},$$

where $\xi=(\xi_i)_{i=1}^n=(\bar x-\tilde x)/(\bar t-\tilde t)$, $|x|=(\sum_{i=1}^n x_i^2)^{1/2}$, and ρ, α are positive constants to be specified below.

We consider the oblique cylinder $Q=\{(t,x):\tilde t\leqq t\leqq\bar t,\psi\geqq 0\}$ with centers of the bases at $(\tilde t,\tilde x)$, $(\bar t,\bar x)$ and radius ρ which is chosen so that $Q\subset S_D^-(t^0,x^0)$ and $u(t,x)<v(t,x)$ on the base B situated on the plane $t=\tilde t$. Introducing the function $w(t,x)=v(t,x)-\varepsilon\phi(t,x)$, $\varepsilon>0$, and using (2), (3), (5), we obtain the following inequality:

(7) $$F(t,x,u,u_tu_x,u_{xx})\geqq F(t,x,w,w_t,w_x,w_{xx})+\varepsilon f,$$

where

$$\begin{aligned} f=&-L_0\phi-4L_1\psi e^{-\alpha(t-\tilde t)}\Big|\sum_{i=1}^n\xi_i(x_i-\tilde x_i-\xi_i(t-\tilde t))\Big| \\ &-L_2\sum_{i=1}^n|\phi_{x_i}|-4nL_3\psi e^{-\alpha(t-\tilde t)} \\ &+8\kappa|x-\tilde x-\xi(t-\tilde t)|^2e^{-\alpha(t-\tilde t)}+\tau\alpha\psi^2e^{-\alpha(t-\tilde t)}. \end{aligned}$$

From the inequality

(8) $$f\geqq e^{-\alpha(t-\tilde t)}[(\tau\alpha-L_0)\psi^2-4\{2\kappa+L_1\rho\sum_{i=1}^n|\xi_i|+n(L_2+L_3)\}\psi+8\kappa\rho^2],$$

it follows that $f>0$ for all sufficiently large $\alpha>0$. Consequently from (7) and (8), we obtain

$$F(t,x,w,w_t,w_x,w_{xx})<F(t,x,u,u_t,u_x,u_{xx}) \text{ in } Q.$$

We choose ε so that $u(t,x)\leqq w(t,x)$ on B. Since this inequality also holds on the lateral surface of Q, we can apply a modified version of a lemma of Nagumo and Simoda [5] to conclude that $u(t,x)\leqq w(t,x)$ in Q. Since $\phi(\bar t,\bar x)>0$, we have $u(\bar t,\bar x)<v(\bar t,\bar x)$ which contradicts the definition of $(\bar t,\bar x)$. Q.E.D.

Now, we shall give an extension of Besala's maximum principle in unbounded regions [2]. Our result also extends a recent result of Chen, Kuroda and Kusano [3] for linear parabolic equations with unbounded coefficients. Let $\varDelta=[0,T]\times\Omega$ be a cylindrical region in

R^{n+1}, where Ω is an unbounded domain in R^n with boundary $\partial\Omega$ and closure $\bar{\Omega}$. We denote by $\partial_p\Delta$ the set $\{[0,T]\times\partial\Omega\}\cup\{(t=0)\times\Omega\}$, the parabolic boundary of Δ.

Theorem 2. *Assume that $F(t,x,z,p,Q,R)$ belongs to the class $\mathcal{P}(\bar{\Delta})$ and satisfies the inequalities:*

$$(9)\qquad \begin{aligned}&F(t,x,z+y,p,Q,R)-F(t,x,z,p,Q,R)\\ &\quad\leqq L_1[\log(|x|^2+1)+1]^{\lambda}(|x|^2+1)^{\mu}y,\end{aligned}$$

$$(10)\qquad \begin{aligned}&|F(t,x,z,p,Q,R)-F(t,x,z,p,\tilde{Q},\tilde{R})|\leqq L_2(|x|^2+1)^{1/2}\sum_{i=1}^{n}|q_i-\tilde{q}_i|\\ &\quad+L_3[\log(|x|^2+1)+1]^{-\lambda}(|x|^2+1)^{1-\mu}\sum_{i,j=1}^{n}|r_{ij}-\tilde{r}_{ij}|\end{aligned}$$

for all $(t,x)\in\bar{\Delta}$ and all $z, y(y\geqq 0), p, Q, \tilde{Q}, R, \tilde{R}$, where $L_1>0$, $L_2\geqq 0$, $L_3>0$, $\mu>0$ and λ are constants.

Let $u(t,x)$ and $v(t,x)$ be continuous in $\bar{\Delta}$, continuously differentiable in Δ once with respect to t and twice with respect to x, and such that

$$(11)\qquad u(t,x)\leqq v(t,x)\qquad on\ \partial_p\Delta,$$

$$(12)\qquad F(t,x,u,u_t,u_x,u_{xx})\geqq F(t,x,v,v_t,v_x,v_{xx})\qquad in\ \Delta,$$

$$(13)\qquad u(t,x)\leqq M_1\exp\{k[\log(|x|^2+1)+1]^{\lambda}(|x|^2+1)^{\mu}\}\qquad in\ \bar{\Delta},$$

$$(14)\qquad v(t,x)\geqq -M_2\exp\{k[\log(|x|^2+1)+1]^{\lambda}(|x|^2+1)^{\mu}\}\qquad in\ \bar{\Delta}$$

for some positive constants M_1, M_2 and k. Then it holds that

$$(15)\qquad u(t,x)\leqq v(t,x)\qquad in\ \bar{\Delta}.$$

Proof. We confine ourselves to the case $\lambda\geqq 0$. The case $\lambda<0$ is treated similarly. Consider the function

$$w(t,x)=\exp\{2ke^{\theta t}[\log(|x|^2+1)+1]^{\lambda}(|x|^2+1)^{\mu}\},\qquad \theta>0$$

and define $g(t,x)$ as follows:

$$\begin{aligned}g(t,x)=&L_1[\log(|x|^2+1)+1]^{\lambda}(|x|^2+1)^{\mu}w+L_2(|x|^2+1)^{1/2}\sum_{i=1}^{n}|w_{x_i}|\\ &+L_3[\log(|x|^2+1)+1]^{-\lambda}(|x|^2+1)^{1-\mu}\sum_{i,j=1}^{n}|w_{x_ix_j}|-\tau|w_t|.\end{aligned}$$

It is a matter of easy computation to verify that

$$g(t,x)\leqq ke^{\theta t}[\log(|x|^2+1)+1]^{\lambda}(|x|^2+1)^{\mu}(H-2\tau\theta)w(t,x)$$

in $(0,\theta^{-1}]\times\Omega$, where

$$H=8nL_3(\lambda+\mu)^2(2ke+1)+4n(\lambda+\mu)(L_2+7L_3)+L_1k^{-1}.$$

Hence, taking $\theta=H/\tau$, we have $g(t,x)<0$ in $(0,\theta^{-1}]\times\Omega$.

Let us consider the functions

$$h^1(t,x)=u(t,x)-M_1h(t,x),\qquad h^2(t,x)=v(t,x)+M_2h(t,x)$$

in $[0,\theta^{-1}]\times\bar{\Omega}_\rho$, where $\Omega_\rho=\Omega\cap\{|x|<\rho\}$, $\rho>0$, and

$$h(t,x)=w(t,x)\exp\{-k[\log(\rho^2+1)+1]^{\lambda}(\rho^2+1)^{\mu}\}.$$

Then, using (9) and (10), we have the inequalities:

$$\begin{aligned}&F(t,x,u,u_t,u_x,u_{xx})\\ &\quad\leqq F(t,x,h^1,h^1_t,h^1_x,h^1_{xx})+L_1[\log(|x|^2+1)+1]^{\lambda}(|x|^2+1)^{\mu}M_1h\\ &\qquad-\tau M_1|h_t|+L_2(|x|^2+1)^{1/2}M_1\sum_{i=1}^{n}|h_{x_i}|\end{aligned}$$

$$+L_3[\log(|x|^2+1)+1]^{-\lambda}(|x|^2+1)^{1-\mu}M_1\sum_{i,j=1}^{n}|h_{x_ix_j}|,$$

$$F(t,x,v,v_t,v_x,v_{xx})$$
$$\geqq F(t,x,h^2,h_t^2,h_x^2,h_{xx}^2)-L_1[\log(|x|^2+1)+1]^{\lambda}(|x|^2+1)^{\mu}M_2h$$
$$+\tau M_2|h_t|-L_2(|x|^2+1)^{1/2}M_2\sum_{i=1}^{n}|h_{x_i}|$$
$$-L_3[\log(|x|^2+1)+1]^{-\lambda}(|x|^2+1)^{1-\mu}M_2\sum_{i,j=1}^{n}|h_{x_ix_j}|,$$

from which, in view of (12), we see that

$$F(t,x,h^2,h_t^2,h_x^2,h_{xx}^2)-F(t,x,h^1,h_t^1,h_x^1,h_{xx}^1)$$
$$\leqq(M_1+M_2)\exp\{-k[\log(\rho^2+1)+1]^{\lambda}(\rho^2+1)^{\mu}\}g(t,x)<0$$

in $(0,\theta^{-1}]\times\Omega_\rho$. From (11), (13), (14) and a lemma of Nagumo and Simoda [5], it follows that $h^1(t,x)\leqq h^2(t,x)$ in $[0,\theta^{-1}]\times\bar{\Omega}_\rho$.

Let (t^*,x^*) be an arbitrary point of $[0,\theta^{-1}]\times\bar{\Omega}$. Then (t^*,x^*) is contained in $[0,\theta^{-1}]\times\bar{\Omega}_\rho$ for all sufficiently large ρ, and we have $h^1(t^*,x^*)\leqq h^2(t^*,x^*)$. Letting $\rho\to\infty$, we see that the inequality $u(t,x)\leqq v(t,x)$ holds throughout $[0,\theta^{-1}]\times\bar{\Omega}$. If $\theta^{-1}<T$, then it suffices to repeat the above procedure a finite number of times to arrive at the required conclusion (15). Q.E.D.

Remark. It is easy to extend our results to the weakly coupled parabolic system

$$F^\alpha(t,x,(u^\beta)_{\beta=1}^N,u_t^\alpha,u_x^\alpha,u_{xx}^\alpha)=0,\qquad \alpha=1,\cdots,N.$$

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