168. The Extension of Darboux's Method to Systems in Involution of Partial Differential Equations of Arbitrary Order in Two Independent Variables

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0. Introduction. Darboux's method of integrating a single partial differential equation of the second order in two independent variables is such that for those equations to which the method may be successfully applied, the solution of Cauchy's problem can be reduced to the integration of a system of ordinary differential equations (cf. E. Goursat [8], A. R. Forsyth [7]). The main aim of our investigation is the extension of Darboux's method to systems in involution of partial differential equations of arbitrary order with one unknown function of two independent variables by applying the theory of differential systems due to E. Cartan (E. Cartan [1]–[5], E. Goursat [9]).

Darboux's method is summarized, from our standpoint, as follows. Consider a single differential equation with an unknown function $z(x, y)$ of two independent variables x, y

$$F(x, y, z, p, q, r, s, t)=0, \tag{1}$$

where $p=\partial z/\partial x$, $q=\partial z/\partial y$, $r=\partial^2 z/\partial x^2$, $s=\partial^2 z/\partial x\partial y$, $t=\partial^2 z/\partial y^2$.

The differential equation (1) can be represented by the differential system

$$\begin{cases} F=0, \quad dF=0, \quad \varpi=dz-pdx-qdy=0, \\ \varpi_1=dp-rdx-sdy=0, \quad \varpi_2=dq-sdx-tdy=0. \end{cases} \tag{2}$$

If $u(x, y, z, p, q, r, s, t)$ is an integral, independent of F, of one of the characteristic systems of the equation (1), then the differential system

$$F=0, \quad u=0, \quad dF=0, \quad du=0, \quad \varpi=0, \quad \varpi_1=0, \quad \varpi_2=0$$

has one-dimensional (Cauchy's) characteristics. Therefore if there exists one more integral of order two, independent of F and u, of the same characteristic system, then Cauchy's problem can be solved by integrating a system of ordinary differential equations. When neither characteristic systems of (1) have two independent integrals of order two, we prolong the system (2). The similar argument implies that, if one of the characteristic systems has two independent integrals (invariants) of possibly higher order, then we can solve Cauchy's problem by integrating a system of ordinary differential equations. This argument can be applied also to a single equation of higher order (cf. E. Goursat [8], § 213).

From this standpoint, we extend Darboux's method to systems in involution of partial differential equations in two independent variables. To do this, we must explicitely write down their characteristic systems and calculate the number of their characteristic systems. Various results obtained already concerning systems in involution and Darboux's method (cf. E. Goursat [8], A. R. Forsyth [7]) follow from our results as corollaries, and our investigation more intrinsically shows the reason why Darboux's method succeeds. All functions which occur in this note are assumed to be analytic though not all the arguments require this assumption. The details of this note will be published elsewhere.

1. Systems in involution. Let us consider the system of partial differential equations of order m with an unknown function $z(x, y)$ of two independent variables x, y

$$(\mathrm{S}_m) \qquad f_\alpha(x, y, z, \cdots, p_{j,k}, \cdots)=0 \qquad (\alpha=1, 2, \cdots, q),$$

where $p_{j,k}=\partial^{j+k}z(x, y)/\partial x^j \partial y^k$. We shall denote by s_m the set of differential equations exactly of order m in S_m:

$$(s_m) \qquad F_\alpha(x, y, z, \cdots, p_{j,k}, \cdots)=0 \qquad (\alpha=1, 2, \cdots, r).$$

The system of equations S_m defines an analytic variety $\mathcal{V}(\mathrm{S}_m)$ in the space of the variables x, y, z, $p_{j,k}$ $(1\leqq j+k\leqq m)$. We assume that at each point on $\mathcal{V}(\mathrm{S}_m)$ the rank of Jacobian matrix of the functions $f_1, f_2, \cdots, f_q$ with respect to the variables $x, y, z, p_{j,k}$ $(1\leqq j+k\leqq m)$ is equal to the co-dimension of the variety $\mathcal{V}(\mathrm{S}_m)$.

The system S_m can be represented by the differential system

$$(\Omega(\mathrm{S}_m)) \quad \begin{cases} f_\alpha=0, \quad df_\alpha=0 \ (\alpha=1, 2, \cdots, q), \quad dz-p_{1,0}dx-p_{0,1}dy=0, \\ dp_{j,k}-p_{j+1,k}dx-p_{j,k+1}dy=0 \ (1\leqq j+k\leqq m-1). \end{cases}$$

Definition. S_m is said to be *in involution* if $\Omega(\mathrm{S}_m)$ is in involution with respect to the variables x, y. We say that S_m is *complete* if all the differential equations of orders at most m which are algebraic consequences of the equations obtained by differentiating once the equations of S_m with respect to x and to y are algebraic consequences of the equations of S_m.

We shall denote by s_{m+1} the set of differential equations of order $m+1$ obtained by differentiating once the equations of s_m with respect to x and to y. We shall call rank $\partial(F_1, F_2, \cdots, F_r)/\partial(p_{m,0}, p_{m-1,1}, \cdots, p_{0,m})$ (on $\mathcal{V}(\mathrm{S}_m)$) the *rank* of s_m. The rank of s_{m+1} is defined similarly. Applying E. Cartan's criterion of involution (E. Cartan [2], [5]), we have the following theorem.

Theorem I. *The system* S_m *is in involution if and only if* S_m *is complete and the rank of* s_{m+1} *is greater than the rank of* s_m *by one.*

(The systems of differential equations considered by G. Cerf ([6], p. 329) are systems in involution in our sense.)

The *characteristic equation* of a single differential equation $F=0$ of order n is, by definition (cf. I. G. Petrovskii [11], § 3, E. Goursat [8], § 209),

$$F^0 dy^n - F^1 dy^{n-1}dx + \cdots + (-1)^n F^n dx^n = 0, \quad \text{where} \quad F^j = \partial F / \partial p_{n-j,j}.$$

The direction of the line through a point (x^0, y^0) in the (x, y)-space defined by $\xi_0 dy - \xi_1 dx = 0$ is called the *characteristic direction* of $F=0$ at the point $p^0 = (x^0, y^0, z^0, p^0_{j,k}\ (1 \leqq j+k \leqq n))$ if $(dy, dx) = (\xi_1, \xi_0)$ is a root of the characteristic equation of $F=0$ at p^0. We define the *characteristic direction of the system* S_m *in involution* at a point on the variety $\mathcal{V}(S_m)$ as the common characteristic direction of all the differential equations of S_m.

Theorem II. *The number of the characteristic directions of* S_m *at each point on* $\mathcal{V}(S_m)$ *is equal to the character of order one of the differential system* $\Omega(S_m)$.

This theorem implies that the number of arbitrary functions (of one argument), on which the general integral of S_m depends, is equal to the number of the characteristic directions of S_m on $\mathcal{V}(S_m)$. It is remarked that the number of characteristic directions is counted with their multiplicities.

2. The extension of Darboux's method. Hereafter we always assume that the system S_m is in involution. Let us prolong the differential system $\Omega(S_m)$ by E. Cartan's total prolongation p. By the theorem due to E. Cartan [2] and Y. Matsushima [10], $p^{n-m}\Omega(S_m)$ is in involution with respect to x, y $(n \geqq m)$. If we explicitely write down the condition that a linear integral element of $p^{n-m}\Omega(S_m)$ is singular, then we obtain a set of differential systems of degree one, which correspond to the characteristic directions of S_m. These differential systems are called the *characteristic systems of order* n of S_m. We shall denote by $C^n(\lambda)$ the characteristic system of order n of S_m corresponding to a characteristic direction defined by $\lambda_0 dy - \lambda_1 dx = 0$, $\lambda = \lambda_1/\lambda_0$.

A function of the variables $x, y, z, p_{j,k}$ $(1 \leqq j+k \leqq n)$ is called a function of *the elements of contact of order* n. An *invariant* of the characteristic system $C^n(\lambda)$ is by definition a function u of the elements of contact of order n such that the equation $du=0$ is a consequence of the equations of $C^n(\lambda)$. A function u of the elements of contact of order n such that du does not vanish in consequence of the equation $u=0$ is called a *relative invariant* of $C^n(\lambda)$ if $du=0$ is a consequence of the equations of $C^n(\lambda)$ and $u=0$. A function of the elements of contact of order n is an invariant of $C^n(\lambda)$ if and only if it is an invariant of $C^{n+1}(\lambda)$. Thus we may forget the order of the characteristic system.

We shall denote by $C(\lambda)$ the characteristic system without specifying the order.

It is well-known that the prolongation p for a system of differential equations is defined, corresponding to the total prolongation p of a differential system. From S_m and h given differential equations $u_1=0$, $u_2=0, \cdots, u_h=0$ respectively of order $n_1, n_2, \cdots, n_h$ ($\geqq m$), we construct, for each integer $n \geqq \max\{n_i\,;\,1\leqq i\leqq h\}$, a system of differential equations

$$(S_n(u_1, u_2, \cdots, u_h)) \quad \begin{cases} & \partial_x^j \partial_y^k u_1 = 0 \qquad (0\leqq j+k\leqq n-n_1), \\ p^{n-m}S_m, & \cdots\cdots\cdots \\ & \partial_x^j \partial_y^k u_h = 0 \qquad (0\leqq j+k\leqq n-n_h), \end{cases}$$

where ∂_x and ∂_y express the total derivatives respectively with respect to x and to y. We say that a differential equation of order $n\geqq m$ is *independent* of the system S_m if it is not reduced to an equation of order lower than n in consequence of the equations of $p^{n-m}S_m$. We can prove the following theorem.

Theorem III. *Let u be a function of the elements of contact of order $n\geqq m$ such that the equation $u=0$ of order n is independent of S_m. Then u is a relative invariant of a characteristic system of S_m if and only if the system $S_n(u)$ is in involution.*

The following is in a certain sense a generalization of the above Theorem III.

Theorem IV. *Let $C(\lambda_1), C(\lambda_2), \cdots, C(\lambda_h)$ be h different characteristic systems of the system S_m and let $u_1, u_2, \cdots, u_h$ be relative invariants respectively of order n_1 of $C(\lambda_1)$, of order n_2 of $C(\lambda_2), \cdots$, of order n_h of $C(\lambda_h)$, where $n_1, n_2, \cdots, n_h \geqq m$. If each of the equations $u_1=0$, $u_2=0, \cdots, u_h=0$ is independent of S_m, then the system $S_N(u_1, u_2, \cdots, u_h)$ is in involution, where $N=\max\{n_i\,;\,1\leqq i\leqq h\}$, and the number of the characteristic directions of $S_N(u_1, u_2, \cdots, u_h)$ on $\mathcal{V}(S_N(u_1, \cdots, u_h))$ is equal to the number of the characteristic directions of S_m on $\mathcal{V}(S_m)$ minus h.*

Furthermore the following theorem is valid.

Theorem V. *If the system S_m has only one characteristic direction at each point on $\mathcal{V}(S_m)$, then the differential system $\Omega(S_m)$ has one-dimensional (Cauchy's) characteristics and the characteristic system of $\Omega(S_m)$ is given by the characteristic system (of order m) of S_m corresponding to the only one characteristic direction of S_m.*

Now Cauchy's problem for the system S_m can be interpreted as follows: "For any given non-characteristic integral curve of the differential system $\Omega(S_m)$, find the two-dimensional integral manifold of $\Omega(S_m)$ passing through that curve."

In treating this problem, we must distinguish the following three cases. Let η be the number of the characteristic directions of S_m at each point on $\mathcal{V}(S_m)$.

1°) When $\eta=0$, S_m is completely integrable and for any given point on $\mathcal{V}(S_m)$ there exists one and only one integral of S_m passing that point.

2°) When $\eta=1$, the solution of Cauchy's problem for S_m can be reduced to the integration of a system of ordinary differential equations.

3°) When $\eta>1$, generally Cauchy's problem for S_m cannot be solved by integrating a system of ordinary differential equations. However for certain systems of partial differential equations, we have a method of integration which is an extension of Darboux's method. In fact the following is valid.

"If $\eta-1$ different characteristic systems of S_m have respectively two independent invariants which are independent of S_m, then the solution of Cauchy's problem can be reduced to the integration of a system of ordinary differential equations."

This is proved by using the above theorems.

References

[1] E. Cartan: Sur l'intégration des systèmes d'équations aux différentielles totales. Ann. Sci. École Norm. Sup., 3e série, **18**, 241–311 (1901).
[2] ——: Sur la structure des groupes infinis de transformations. Ann. Sci. École Norm. Sup., 3e série, **21**, 153–206 (1904).
[3] ——: Les systèmes de Pfaff a cinq variables et les équations aux dérivées partielles du second ordre. Ann. Sci. École Norm. Sup., 3e série, **27**, 109–192 (1910).
[4] ——: Leçons sur les invariants intégraux. Hermann, Paris (1922).
[5] ——: Les systèmes différentiels extérieurs et leurs application géométriques. Hermann, Paris (1945).
[6] G. Cerf: Sur les transformations des équations aux dérivées partielles d'ordre quelconque à deux variables indépendentes. J. de Math., 7e série, **4**, 309–412 (1918).
[7] A. R. Forsyth: Theory of Differential Equations, Part IV, Vol. VI. Cambrige Univ. Press, London (1906).
[8] E. Goursat: Leçons sur l'intégration des équations aux dérivées partielles du second ordre à deux variables indépendentes, Tom. II. Hermann, Paris (1898).
[9] ——: Leçons sur le problème de Pfaff. Hermann, Paris (1922).
[10] Y. Matsushima: On a theorem concerning the prolongation of a differential system. Nagoya Math. J., **6**, 1–16 (1953).
[11] I. G. Petrovskii: Lectures on Partial Differential Equations (2nd ed.). Gos. Izd. Tekh.-Teor. Lit., Moscow (1953).