# 14. *On the Asymptotic Behavior of Resolvent Kernels and Spectral Functions for Some Class of Hypoelliptic Operators*

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(Comm. by Kinjirô KUNUGI, M. J. A., Jan. 12, 1974)

1. **Introduction.** For hypoelliptic operators with constant coefficients studies on asymptotic behavior of their spectral functions were done by Nilsson [10], Gorčakov [6] and Friberg [4] (cf. [15]). For the case of operators with variable coefficients Nilsson [11] has studied this problem for formally hypoelliptic operators and Smagin [12] has done that for some class of hypoelliptic operators for which a complex power can be defined. In this paper we shall anounce some results on that problem and asymptotic distribution of eigenvalues for the case of variable coefficients by a method of pseudo-differential operators (cf. [7], [8]). Let $P=P(x,D)=\sum_{|\alpha|\le m} a_\alpha(x)D^\alpha$ be a formally self-adjoint linear partial differential operator with its domain $C_0^\infty(\Omega)$, where $x=(x_1,\cdots,x_n)$ is a point of real $n$-space $R_x^n$, $\alpha=(\alpha_1,\cdots,\alpha_n)$ is a multi-index of which length $|\alpha|=\alpha_1+\cdots+\alpha_n$ and $D^\alpha$ or $D_x^\alpha=(-i\partial/\partial x_1)^{\alpha_1}\cdots(-i\partial/\partial x_n)^{\alpha_n}$. The coefficients $a_\alpha(x)$ are supposed to be in $\mathscr{B}(\Omega)$ in the notation of L. Schwarz for an open set $\Omega$ in $R_x^n$. For $\xi\in R^n$ we denote $|\xi|=(\xi_1^2+\cdots+\xi_n^2)^{1/2}$, $\langle\xi\rangle=1+|\xi|$ and $\xi^\alpha=\xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n}$. For $P(x,\xi)$ $\sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$ we set $P_{(\beta)}^{(\alpha)}(x,\xi)=D_\xi^\alpha(iD_x)^\beta P(x,\xi)$.

2. **A class of hypoelliptic operators, theorems.** We assume the followings on $P(x,\xi)$: this is written in the sum $P(x,\xi)=p_0(x,\xi)+p_1(x,\xi)$ and for any $x\in\Omega$ and $\alpha$ and $\beta$ there exist positive constants $C_{x,\alpha,\beta}$, $C_x$ and $A_x$ such that

$$|p_{0(\beta)}^{(\alpha)}(x,\xi)|\le C_{x,\alpha,\beta}|p_0(x,\xi)|^{1-\rho|\alpha|+\delta|\beta|} \tag{2.1}$$

$$|p_{1(\beta)}^{(\alpha)}(x,\xi)|\le C_{x,\alpha,\beta}|p_0(x,\xi)|^{1-\rho(|\alpha|+1)+\delta(|\beta|+1)} \tag{2.1$'$}$$

for $|\xi|\ge A_x$, where $\rho$ and $\delta$ are some constants depending only on $P(x,\xi)$ and satisfying $0\le\rho<\delta\le 1/m$, and

$$|p_0(x,\xi)|\ge C_x|\xi|^{m'},\quad 0<m'\le m\quad \text{for } |\xi|\ge A_x, \tag{2.2}$$

$$m'>n. \tag{2.3}$$

We remark that (2.3) can be removed by considering a power of $P(x,D)$. We assume further that $C_{x,\alpha,\beta}$, $C_x$ and $A_x$ are bounded when $x$ is in a compact subset of $\Omega$. We consider the case in which $p_0(x,\xi)$ is taken real because of the self-adjointness of $P(x,D)$, and assume $p_0(x,\xi)\to+\infty$ as $|\xi|\to\infty$. We have proved in [13] the following:

**Theorem 1.** *Every differential operator $P(x, D)$ satisfying* (2.1) (2.2) *is hypoelliptic in $\Omega$.*

We denote $\tilde{P}=\tilde{P}(x, D)$ a self-adjoint realization in $L^2(\Omega)$ of $P$ with the domain $C_0^\infty(\Omega)$ and $(\tilde{P}-\lambda)^{-1}$ the resolvent of $P$ for $\lambda \notin R_+ = \{\lambda : \lambda \in R^1, \lambda > 0\}$ which is definable as a bounded operator in $L^2(\Omega)$ by Theorem 12.7 of [1], p. 184.

**Theorem 2.** *Under* (2.1) ~ (2.3), $(\tilde{P}-\lambda)^{-1}$ *has an integral kernel* $G_\lambda(x, y)$ of continuous Carleman type (cf. [2] p. 5) and there are $g_j(x, y, \lambda)$ $j=0, 1, 2, \cdots$, in $C(\Omega \times \Omega)$ such that asymptotically

(2.4) $$|G_\lambda(x, y)-\textstyle\sum_{j=0}^{k} g_j(x, y, \lambda)| \leq C(-\lambda)^{-s(k, a)}$$

uniformly for $|x-y|(-\lambda)^{a\delta} \leq d$ and $(x, y) \in K \times K$, $K \subset\subset \Omega$, where

$$s(a, k)=a(\rho-\delta)(k+1)+\max\{a, a(\rho-\delta)(k+1)\}, \quad g_0(x, x, \lambda) = (2\pi)^{-n}\int_{R^n} (p_0(x, \xi)-\lambda)^{-1} d\xi$$

and the estimates $|g_j(x, y, \lambda)| \leq C(-\lambda)^{-a(\rho-\delta)k-1}$ hold for any a such that $0 < a \leq 1-n/m'$. The contact operator $P^0=P(x^0, D)$ at $x^0$ satisfies also (2.1) ~ (2.3) and hence hypoelliptic by Theorem 1. The existence of the spectral functions $e(x, y, t)$ and $e^0(x, y, t)$ of $\tilde{P}$ and $\tilde{P}^0$ respectively is proved under (2.1) ~ (2.2) and we have by adding (2.3)

$$G_\lambda(x, y)=\int_0^\infty (t-\lambda)^{-1} d_t(x, y, t), \qquad \text{(cf. [2] pp. 5–7).}$$

By the theorem of Nilsson [10] Theorem 1 (p. 530) it holds that for the spectral function of $P^0$

$$C^{-1}t^b (\log t)^r \leq e^0(x, x, t) \leq Ct^b (\log t)^r$$

where $b$ and $r$ are a positive and a non-negative integer respectively.

**Theorem 3.** *Assume $b \geq n/m'-(1-n/m')(\rho-\delta)$ adding the same assumption of Theorem 2. Then we have for the spectral function of $\tilde{P}$*

(2.5) $$e(x, y, t)=o(1) \qquad \text{for } x \neq y, \quad t \to +\infty$$

(2.5)′ $$C^{-1}t^b (\log t)^r \leq e(x, x, t) \leq Ct^b (\log t)^r (t > c)$$

*for some positive number $b$ and non-negative integer $r$.*

**Theorem 4.** *Assume furthermore that $(\tilde{P}+iI)^{-1}$ is a compact operator on $L^2(\Omega)$ and $V_{p_0}(t, \Omega)=\int_\Omega \int_{p_0(x,\xi)<t} d\xi dx$, then we have*

(2.6) $$N(t)=\textstyle\sum_{\lambda_j<t} 1 = CV_{p_0}(t, \Omega)+o(1)V_{p_0}(t, \Omega) \qquad t \to +\infty.$$

**3. Outline of proof of theorems.** The proof of the theorems is obtained from following series of lemmas. First we construct a parametrix $E_k$ of $P(x, D)-\lambda$ $(\lambda \notin R_+)$ with which we compare $G_\lambda(x, y)$. Let $q_j=q_j(x, \xi, \lambda)$, $j=0, 1, 2, \cdots$, be defined successively in the following: for $p_\lambda=p_0(x, \xi)-\lambda$ $(\lambda \notin R_+)$

(3.1) $$q_0=1/p_\lambda$$

(3.2) $$q_j=-(1/p_\lambda)(p_1 q_{j-1}+\textstyle\sum_{|\gamma|+l=j,\ l<j} P^{(\gamma)} q_{l(\gamma)}) \qquad \text{for } |\xi| \geq A_x.$$

Let $Q_j(x, y, \lambda)$ be the distribution kernel (cf. [9] pp. 140–1) corresponding to the distribution:

$$C_0^\infty(\Omega) \ni u \to (2\pi)^{-n} \int_{R^n} q_j(x, \xi, \lambda) e^{i\langle x, \xi \rangle} \hat{u}(\xi) d\xi,$$

where $\hat{u}(\xi)$ is the Fourier transform of $u$. Remarking that $p_\lambda^{-1} \leq 6(-\lambda)^{-a} (|p_0|+1)^{a-1}$ for $|\xi| \geq A_x$, and $\lambda < l(x) = \min. (-2m(x), -1)$, where $m(x) = \sup_{|\xi| \leq A_x} |p_0(x, \xi)|$, and the number a should be taken in the interval $0 < a < 1 - n/m'$ for the integrability of $(|p_0|+1)^{-1+a}$, we have

**Lemma 1.** *Assume* (2.1) ~ (2.2).

(1°) *For* $j > (1/(\rho-\delta))\{n/(1-a)m' - 1\}$, $0 < a \leq 1 - n/m'$, *we have* $Q_j(x, y, \lambda) \in C(\Omega \times \Omega)$ *and* $|Q_j(x, y, \lambda)| \leq C(-\lambda)^{-a\{(\rho-\delta)j+1\}}$ *for* $\lambda < l(x)$, *uniformly on* $(x, y) \in K \times K : K \subset \subset \Omega$.

(2°) *For* $|x-y|(-\lambda)^{a\delta} \geq d > 0$ *we have* $Q_j(x, y, \lambda) \in C^\infty(\Omega \times \Omega)$ *and for any* $\kappa \geq 0$ $|D_x^{\gamma'} D_y^{\gamma''} Q_j(x, y, \lambda)| \leq C(-\lambda)^{-a\{(\rho-\delta)\kappa+1-\delta|\gamma'|\}}$ *for* $\lambda < l(x)$, *uniformly on* $(x, y) \in K \times K : K \subset \subset \Omega$.

Let $F_k(x, y, \lambda)$ be the distribution kernel corresponding to the distribution:

$$C_0^\infty(\Omega) \ni u \to (2\pi)^{-n} \int_{R^n} \sum_{j=0}^k q_j(x, \xi, \lambda) e^{i\langle x, \xi \rangle} \hat{u}(\xi) d\xi.$$

We define the distribution $E_k$ of which kernel is a cutting of $F_k(x, y, \lambda)$: $E_k(x, y, \lambda) = \varphi_0(|x-y|(-\lambda)^{a\delta}) F_k(x, y, \lambda)$, where $\varphi_0(x) \in C_0^\infty(|x| < d')$.

**Lemma 2.** *Assume* (2.1) ~ (2.3). *Then we have* $(P(x, D) - \lambda) E_k = \delta(x-y) + \omega_k$, *and* $|\omega_k(x, y, \lambda)| \leq C(-\lambda)^{-a(\rho-\delta)(k+1)}$, *where* $\omega_k(x, y, \lambda)$ *is the kernel of the distribution* $\omega_k$ *and to be as smooth as we wish.*

The existence of the resolvent kernel $G_\lambda(x, y)$ of continuous Carleman type is derived from $(\tilde{P} - \lambda)^{-1} \in H_{m'}^{\mathrm{loc}}(\Omega)$ by (2.3) and we have the following lemma from which Theorem 2 is immediate.

**Lemma 3.** *Under* (2.1) ~ (2.3) *we have the estimate*

(3.3) $$|G_\lambda(x, y) - E_k(x, y, \lambda)| \leq C(-\lambda)^{-s(k,a)} \quad \lambda < l(x),$$

*uniformly on* $(x, y) \in K \times K : K \subset \subset \Omega$, *where* $s(k, s)$ *is that in the statement of Theorem* 2, *and this is estimated by any power of* $(-\lambda)$ *for* $x \neq y$.

Let $E_k^0(x, y, \lambda)$ and $G_\lambda^0(x, y)$ are the parametrix and the resolvent kernel of the contact operator $P^0$ respectively. We shall use the following further result of Nilsson [10]:

$$(d/dt) e^0(x^0, x^0, t) = o(1) t^{b-1} (\log t)^r \qquad t \to +\infty.$$

We can prove a similar statement as (3.3) for $G_\lambda^0(x, y)$ and $E^0(x, y, \lambda)$ because of $P(x^0, D)$ having (2.1) ~ (2.2). Using these results and the fact $E(x^0, x^0, \lambda) = E^0(x^0, x^0, \lambda) = g_0(x^0, x^0, \lambda)$, we have

**Lemma 4.** *There is a positive constant* $c$ (*actually* $c \geq a(\rho-\delta)$) *such that*

(3.4) $$|G_\lambda(x^0, x^0) - G_\lambda^0(x^0, x^0)| \leq O(1)(-\lambda)^{-c}(-\lambda)^{b-1} (\log(-\lambda))^r \qquad \lambda \to -\infty.$$

To obtain (2.5)′ in Theorem 3 from (3.4) we may use Tauberian theorem of Ganelius [5] Theorem 2, p. 217, and have

$$|e(x^0, x^0, t)-e^0(x^0, x^0, t)|\leq O(1)t^b (\log t)^{r-1} \qquad t\to+\infty$$

from which (2.5)′ is immediate. (2.5) can be derived from (2°) of Lemma 2.

When $(\tilde{P}+iI)^{-1}$ is a compact operator, $e(x, y, t)=\sum_{\lambda_j\leq t}\overline{\varphi_j(x)}\varphi_j(y)$ where $\varphi_j$, $j=0, 1, 2, \cdots$, is an orthonormal set of eigenfunctions with eigenvalues $\lambda_j$. Therefore under the assumptions of Theorem 4 we have (2.6) by integrating (2.5)′ and noting that

$$e^0(x^0, x^0, t)=(2\pi)^{-n}\int_{p^0(x^0,\xi)<t} d\xi.$$

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